Math 257/316, Midterm 2, Section 103

November 2009

(1) We first find a steady-state solution $w(x)$:

$$0 = w'' - w$$
$$\Rightarrow \quad w'' - w = 0$$
$$\Rightarrow \quad w(x) = A\cosh(x) + B\sinh(x)$$

for some constants, $A, B \in \mathbb{R}$

Apply BCs:

$$w'(0) = 0 \quad \Rightarrow \quad 0 = \cancel{A\sinh(0)} + B\cosh(0)$$
$$\Rightarrow \quad B = 0$$
$$w(1) = \cosh(1) \quad \Rightarrow \quad \cosh(1) = A\cosh(1)$$
$$\Rightarrow \quad A = 1$$
$$\Rightarrow \quad \boxed{w(x) = \cosh(x)}$$

Let $\quad v(x,t) = u(x,t) - w(x)$

Then $\quad v_t = v_{xx} - v$

$$v_x(0,t) = 0$$
$$v(1,t) = 0$$
$$v(x,0) = \cancel{\cosh(x)} + \frac{\pi x}{2} - \cancel{\cosh(x)}$$
$$= \frac{\pi x}{2}$$

Separation of variables:

write $\quad v(x,t) = X(x)\,T(t)$

$$\Rightarrow \quad XT' = X''T - XT$$
$$\Rightarrow \quad \frac{T'}{T} = \frac{X''}{X} - 1$$
$$\Rightarrow \quad \frac{T'}{T} + 1 = \frac{X''}{X} = -\lambda^2 \quad \text{for nontrivial solutions}$$

$$X'' + \lambda^2 X = 0$$

$$\Rightarrow \quad X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

Apply BCs:

$$X'(0) = 0 \Rightarrow 0 = -\lambda A \sin(0) + \lambda B \cos(0)$$

$$\Rightarrow B = 0$$

$$X(1) = 0 \Rightarrow 0 = A\cos(\lambda)$$

$$\Rightarrow \boxed{\lambda_n = \left(n+\tfrac{1}{2}\right)\pi}, \quad n = 0, 1, 2, \ldots$$

$$\Rightarrow \boxed{X_n(x) = \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)}$$

$$\frac{T'}{T} + 1 = -\lambda^2$$

$$\Rightarrow \frac{T'}{T} = -(\lambda^2 + 1)$$

$$\Rightarrow T_n' + (\lambda_n^2 + 1) T_n = 0$$

$$\Rightarrow T_n(t) = e^{-(\lambda_n^2 + 1)t}$$

$$\Rightarrow \boxed{T_n(t) = e^{-\left(\left(n+\frac{1}{2}\right)^2\pi^2 + 1\right)t}}$$

Principle of superposition

$$\Rightarrow v(x,t) = \sum_{n=0}^{\infty} c_n e^{-\left(\left(n+\frac{1}{2}\right)^2\pi^2+1\right)t} \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)$$

for constants $c_n \in \mathbb{R}$

Apply IC: $\frac{\pi x}{2} = v(x,0)$

$$= \sum_{n=0}^{\infty} c_n \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)$$

$$\Rightarrow \quad c_n = 2\int_0^1 \frac{\pi x}{2} \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)\, dx$$

$$\Rightarrow \quad c_n = \pi\int_0^1 x \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)\, dx$$

$$\Rightarrow \quad \boxed{c_n = \frac{(-1)^n}{n+\frac{1}{2}} - \frac{1}{\left(n+\frac{1}{2}\right)^2 \pi}}$$

Finally $\quad u(x,t) = w(x) + v(x,t)$

$$\Rightarrow \quad \boxed{u(x,t) = \cosh(x) + \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{n+\frac{1}{2}} - \frac{1}{\left(n+\frac{1}{2}\right)^2 \pi}\right] e^{-\left(\left(n+\frac{1}{2}\right)^2\pi^2+1\right)t} \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)}$$

(2) First look for a simple function satisfying the BCs,

take $w(x,t) = A(t)\,x^2 + B(t)\,x$

due to Neumann BCs.

$$w_x(x,t) = 2A(t)\,x + B(t)$$

Require $w_x(0,t) = 0 \Rightarrow 0 = B(t)$

& $w_x(\frac{\pi}{2},t) = 2t \Rightarrow 2t = \cancel{2}A(t)\frac{\pi}{\cancel{2}}$

$$\Rightarrow A(t) = \frac{2t}{\pi}$$

thus $\boxed{w(x,t) = \frac{2t x^2}{\pi}}$

Let $v(x,t) = u(x,t) - w(x,t)$

$$u_t = \alpha^2 u_{xx}$$

$$\Rightarrow (v+w)_t = \alpha^2 (v+w)_{xx}$$

$$\Rightarrow v_t + \frac{2x^2}{\pi} = \alpha^2 v_{xx} + \frac{4\alpha^2 t}{\pi}$$

$$\Rightarrow v_t = \alpha^2 v_{xx} + \frac{4\alpha^2 t}{\pi} - \frac{2x^2}{\pi}$$

$$v_x(0,t) = 0$$

$$v_x(\tfrac{\pi}{2},t) = 0$$

$$v(x,0) = u(x,0) - w(x,0)$$
$$= 5$$

Neumann BCs with $L = \frac{\pi}{2}$

$\Rightarrow$ eigenfunctions are $\cos(2nx)$, $n = 0,1,2,\ldots$

Write $v(x,t) = \dfrac{v_0(t)}{2} + \displaystyle\sum_{n=1}^{\infty} v_n(t)\cos(2nx)$

Substitute into PDE

$$\Rightarrow \frac{v_0'(t)}{2} + \sum_{n=1}^{\infty} v_n'(t)\cos(2nx) = \sum_{n=1}^{\infty} v_n(t)\left(-4\alpha^2 n^2\right)\cos(2nx)$$

$$+ \frac{b_0(t)}{2} + \sum_{n=1}^{\infty} b_n(t)\cos(2nx) \quad \text{—} \ (*)$$

where $$b_n(t) = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \left(\frac{4\alpha^2 t}{\pi} - \frac{2x^2}{\pi}\right) \cos(2nx)\, dx$$

$\underline{n=0}$

$$b_0(t) = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \frac{4\alpha^2 t}{\pi} - \frac{2x^2}{\pi}\, dx$$

$$= \frac{4}{\pi}\left(\frac{4\alpha^2 t x}{\pi} - \frac{2x^3}{3\pi}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{4}{\pi}\left(2\alpha^2 t - \frac{\pi^2}{12}\right)$$

$$= \frac{8\alpha^2 t}{\pi} - \frac{\pi}{3}$$

$\underline{n>0}$

$$b_n(t) = \frac{4}{\pi}\left[\frac{4\alpha^2 t}{\pi}\int_0^{\frac{\pi}{2}} \cos(2nx)\, dx - \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x^2 \cos(2nx)\, dx\right]$$

$$= \frac{4}{\pi}\left[\cancel{\frac{4\alpha^2 t}{\pi}\frac{1}{2n}\sin(2nx)\Big|_0^{\frac{\pi}{2}}} - \frac{2}{\pi}\frac{\pi(-1)^n}{4n^2}\right]$$

$$= \frac{4}{\pi}\frac{(-1)^{n+1}}{2n^2}$$

$$= \frac{2(-1)^{n+1}}{n^2\pi}$$

Matching terms in ~~above~~ (*) that are constant with respect to x gives:

$$\frac{V_0'(t)}{2} = \frac{b_0(t)}{2}$$

$$\Rightarrow V_0'(t) = \frac{8\alpha^2 t}{\pi} - \frac{\pi}{3}$$

$$\Rightarrow \boxed{V_0(t) = \frac{4\alpha^2 t^2}{\pi} - \frac{\pi t}{3} + c_0}$$

for some constant, $c_0 \in \mathbb{R}$

Matching terms in (*) with $\cos(2nx)$ gives

$$V_n'(t) + 4\alpha^2 n^2 V_n(t) = \frac{2(-1)^{n+1}}{n^2\pi}$$

$$\Rightarrow \frac{d}{dt}\left(e^{4\alpha^2n^2t} V_n(t)\right) = \frac{2(-1)^{n+1}}{n^2\pi} e^{4\alpha^2n^2t}$$

(solve the ODE by an integrating factor)

$$\Rightarrow e^{4\alpha^2n^2t} V_n(t) = \frac{2(-1)^{n+1}}{n^2\pi}\frac{1}{4\alpha^2n^2} e^{4\alpha^2n^2t} + c_n$$

for some constant $c_n \in \mathbb{R}$

$$\Rightarrow \boxed{V_n(t) = \frac{(-1)^{n+1}}{2\alpha^2n^4\pi} + c_n e^{-4\alpha^2n^2t}}$$

Apply IC:

$$5 = v(x,0) = \frac{V_0(0)}{2} + \sum_{n=1}^{\infty} V_n(0)\cos(2nx)$$

$$\Rightarrow 5 = \frac{c_0}{2} + \sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{2\alpha^2n^4\pi} + c_n\right]\cos(2nx)$$

$$\Rightarrow \boxed{c_0 = 10}$$

$$\& \quad \boxed{c_n = \frac{(-1)^n}{2\alpha^2n^4\pi}}, \quad n > 0$$

Finally, $u(x,t) = w(x,t) + v(x,t)$

$$\Rightarrow \boxed{u(x,t) = \frac{2t x^2}{\pi} + \frac{1}{2}\left[\frac{4\alpha^2t^2}{\pi} - \frac{\pi t}{3} + 10\right] + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2\alpha^2n^4\pi}\left(1 - e^{-4\alpha^2n^2t}\right)\cos(2nx)}$$