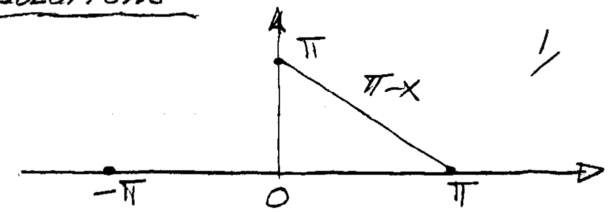
SEC 103 MATH 257/316 MIDTERM 2 SOLUTIONS

1. $$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$$



(a) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left\{\cancel{\pi \frac{\sin(nx)}{n}\Big|_0^{\pi}}\right\} - \frac{1}{\pi}\left[\cancel{x\frac{\sin(nx)}{n}\Big|_0^{\pi}} - \int_0^{\pi} \frac{\sin(nx)}{n}\,dx\right\} = -\frac{\cos(nx)}{\pi n^2}\Big]_0^{\pi} = \frac{1-(-1)^n}{\pi n^2}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\,dx = \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cos(nx)}{n}\right]_0^{\pi} - \frac{1}{\pi}\left\{-x\frac{\cos(nx)}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx\right\}$$

$$= \frac{[1-(-1)^n]}{n} + \frac{(-1)^n}{n} - \frac{1}{\pi}\cancel{\frac{\sin nx}{n^2}}\Big|_0^{\pi} = \frac{1}{n}$$

$$\therefore\ f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{\pi n^2}\cos(nx) + \frac{1}{n}\sin(nx) = S(x)$$

(b) THE SERIES CONVERGES TO $S(0) = \pi/2$.

$$\therefore\ \frac{\pi}{2} = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{\pi n^2}$$

$$\therefore\ \frac{\pi^2}{4} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{n^2}$$

$$\therefore\ \frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

2. $u_t = u_{xx} + x \qquad 0<x<1 \qquad t>0$

$u(0,t) = 0 = u(1,t)$

$u(x,0) = \sin(\pi x) - \frac{1}{6}(x^3 - x)$

WE LOOK FOR A STEADY SOLUTION: $V(x)$:

$$0 = V_{xx} + x \qquad V_x = -\frac{x^2}{2} + A \qquad V(x) = -\frac{x^3}{6} + Ax + B$$

$$V(0) = 0 = B \qquad V(1) = -\frac{1}{6} + A \Rightarrow A = \frac{1}{6} \Rightarrow V(x) = -\frac{1}{6}(x^3 - x)$$

NOW LET $u(x,t) = V(x) + W(x,t)$

$$(\cancel{V(x)}^{0} + W(x,t))_t = W_{xx} + \{\cancel{V_{xx} + x}\}^{0} \Rightarrow W_t = W_{xx}$$

$$0 = u(0,t) = \overset{0}{V(0)} + W(0,t) = W(0,t)$$

SIMILARLY $0 = u(1,t) = V(1) + W(1,t) = W(1,t)$

$$u(x,0) = \cancel{V(x)} + W(x,0) = \sin(\pi x) \underbrace{- \frac{1}{6}(x^3 - x)}_{V(x)} \Rightarrow W(x,0) = \sin(\pi x)$$

LET $W(x,t) = X(x)T(t)$

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 \Rightarrow \qquad X'' + \lambda^2 X = 0 \qquad X = A\cos(\lambda x) + B\sin(\lambda x)$$

$$X(0) = 0 = X(1) \qquad X(0) = A = 0 \quad X(1) = B\sin(\lambda) = 0$$

EIGENVALUES ARE $\lambda_n = n\pi \quad n = 1, 2, \ldots$ EIGENFCN'S $X_n = \sin(n\pi x)$

$$\dot{T}(t) = -\lambda^2 T \Rightarrow T(t) = C_n e^{-\lambda_n^2 t} = C_n e^{-n^2\pi^2 t}$$

$$\therefore \quad W(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

$$W(x,0) = \sin(\pi x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) \Rightarrow \begin{array}{ll} C_n = 0 & n \neq 1 \\ C_1 = 1 & n = 1 \end{array}$$

$$\therefore \quad u(x,t) = \frac{-1}{6}(x^3 - x) + e^{-\pi^2 t}\sin(\pi x)$$

3. $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

$u(r,0) = 0 = u(r,\pi)$

$u(a,\theta) = \sin(2\theta)$

LET $u(r,\theta) = R(r)\,\Theta(\theta)$

$$\frac{r^2(R'' + \frac{1}{r}R')}{R} = -\frac{\Theta''}{\Theta} = \lambda^2$$

$$\left.\begin{array}{l} \Theta'' + \lambda^2\Theta = 0 \\ \Theta(0) = 0 = \Theta(\pi) \end{array}\right\} \quad \begin{array}{l} \Theta = A\cos(\lambda\theta) + B\sin(\lambda\theta) \\ \Theta(0) = A = 0 \quad \Theta(\pi) = B\sin(\lambda\pi) = 0 \quad \lambda\pi = n\pi \; n = 1,2 \end{array}$$

EIGENVALUES ARE $\lambda_n = n \quad n = 1, 2, \ldots$

$\Theta_n = \sin(n\theta)$

$r^2R'' + rR' - \lambda^2 R = 0$ WHICH IS AN EULER EQ.

$R(r) = r^\gamma \Rightarrow \gamma(\gamma - 1) + \gamma - \lambda^2 = \gamma^2 - \lambda^2 = 0 \quad \gamma = \pm\lambda = \pm n.$

$R_n(r) = c_n r^n + d_n r^{-n}$

SINCE $u(r,\theta) < \infty$ AS $r \to 0$ WE REQUIRE $d_n = 0$.

$$u(r,\theta) = \sum_{n=1}^{\infty} c_n r^n \sin(n\theta)$$

NOW $u(a,\theta) = \sin 2\theta = \sum_{n=1}^{\infty} c_n a^n \sin(n\theta)$

$\therefore \quad c_n = 0 \quad n \neq 2$ AND $c_2 = a^{-n}.$

$$\therefore \quad u(r,\theta) = \left(\frac{r}{a}\right)^2 \sin(2\theta)$$