MATH 257/316 MIDTERM 2 SECTIONS 101/102 NOV' 2009.

Q1: $u_t = u_{xx} - u \quad 0<x<1,\ t>0$

$u_x(0,t)=0 \quad u_x(1,t)=\sinh(1)$

$u(x,0)=\cosh(x)-x$

ANSWER: LOOK FOR A STEADY SOLUTION $\omega(x): \omega''-\omega=0,\ \omega'(0)=0,\ \omega'(1)=\sinh(1)$

$\omega=e^{rx} \Rightarrow r^2-1=0 \Rightarrow \omega(x)=A\cosh(x)+B\sinh(x),\ \omega'=A\sinh(x)+B\cosh(x)$

$\omega'(0)=B=0 \quad \omega'(1)=A\sinh(1)=\sinh(1) \Rightarrow A=1$

$\therefore \quad \omega(x)=\cosh(x).$

NOW LET $u(x,t)=\omega(x)+v(x,t)$

$u_t = (\omega(x)+v)_t = \cancel{(\omega_{xx}-\omega)}^{0}+(v_{xx}-v) = u_{xx}-u \Rightarrow v_t = v_{xx}-v$

BC: $0=u_x(0,t)=\cancel{\omega'(0)}^{0}+v_x(0,t) = v_x(0,t) \Rightarrow v_x(0,t)=0$

$\cancel{\sinh(1)}=u_x(1,t)=\omega'(1)+v_x(1,t)=\cancel{\sinh(1)}+v_x(1,t) \Rightarrow v_x(1,t)=0$

$\cancel{\cosh(x)}-x = u(x,0)=\cancel{\cosh(x)}+v(x,0) \Rightarrow v(x,0)=-x$

NOW SEPARATE VARIABLES $v(x,t)=X(x)T(t)$.

$\dot T/T(t)+1 = X''/X(x) = -\lambda^2 \Rightarrow \dot T = -(1+\lambda^2)T \quad T(t)=Ce^{-(1+\lambda^2)t}$

$\left.\begin{array}{l}\lambda\neq 0: X''+\lambda^2 X=0 \\ X'(0)=0=X'(1)\end{array}\right\}$ $X(x)=A\cos\lambda x+B\sin\lambda x \quad X'(x)=-A\lambda\sin\lambda x+B\lambda\cos\lambda x$

$X'(0)=B\lambda=0 \Rightarrow B=0 \quad X'(1)=-A\lambda\sin\lambda=0 \Rightarrow \lambda_n=n\pi \quad n=1,2.$

$\lambda=0$: $X''=0 \Rightarrow X=Ax+B \quad X'(x)=A,\ X'(0)=A=0,\ X_0(x)=1$ IS AN EIGENFUNCTION

EIGENVALUES ARE $\lambda_n=n\pi \quad n=0,1,\ldots \quad X_n\in\{1,\cos(n\pi x)\}$.

$$v(x,t)=\frac{a_0}{2}e^{-t}+\sum_{n=1}^{\infty} a_n e^{-(1+\lambda_n)t}\cos(n\pi x)$$

$$-x = v(x,0) = a_0/2+\sum_{n=1}^{\infty} a_n\cos(n\pi x)$$

$a_0 = 2/1\int_0^1 -x\,dx = 2.(-x^2/2)\big|_0^1 = -1$

$a_n = \left\{\begin{array}{ll} 2/n^2\pi^2 & n \text{ ODD} \\ 0 & n \text{ EVEN}\end{array}\right\}$ GIVEN

$$\therefore\ u(x,t)=\left\{\frac{-1}{2}e^{-t}+\frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{e^{-(1+(2k+1)^2\pi^2)t}}{(2k+1)^2}\cos(n\pi x)\right\}+\cosh(x)$$

Q2: $u_t = \alpha^2 u_{xx} \quad 0 < x < \pi/2,\ t > 0$

$u(0,t) = 0 \quad u_x(\pi/2, t) = t^2/2$

$u(x,0) = 0$

ANSWER: FIND A FUNCTION $\omega(x,t)$ THAT SATISFIES THE BC

TRY $\omega(x,t) = A(t)X + B(t) \qquad \omega(0,t) = B(t) = 0$

NOW $\omega_x(x,t) = A(t) \Rightarrow \omega_x(\pi/2, t) = A(t) = t^2/2 \quad \therefore \quad \omega(x,t) = (t^2 x)/2$

NOW LET $u(x,t) = \omega(x,t) + v(x,t)$

$$u_t = \omega_t + v_t = xt + v_t = \alpha^2(\cancelto{0}{\omega_{xx}} + v_{xx}) \Rightarrow v_t = \alpha^2 v_{xx} - xt$$

BC: $0 = u(0,t) = \cancelto{0}{\omega(0,t)} + v(0,t) \qquad \Rightarrow \quad v(0,t) = 0$

$\cancel{t^2/2} = u_x(\pi/2, t) = \omega_x(\pi/2, t) + v(\pi/2, t) = \cancel{t^2/2} + v(\pi/2, t) \Rightarrow v_x(\pi/2, t) = 0$

$0 = u(x,0) = \cancelto{0}{\omega(x,0)} + v(x,0) \Rightarrow \qquad v(x,0) = 0$

IF WE CONSIDER THE HOMOGENEOUS EQ $v_t = \alpha^2 v_{xx}$ THEN SEPARATION OF VARIABLES YIELDS $\dot{T}/\alpha^2 T = X''/X = -\lambda^2 \qquad T(t) = Ce^{-\lambda^2 t}$

$$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X'(\pi/2) \end{array}\right\} \quad \begin{array}{l} X = A\cos\lambda x + B\sin(\lambda x) \quad X(0) = A = 0 \\ X' = B\lambda\cos\lambda x \quad X'(\pi/2) = B\lambda\cos(\lambda\pi/2) = 0 \end{array}$$

$\therefore$ EIGENVALUES ARE $\lambda_n = (2n+1) \quad n = 0, 1, \ldots \quad X_n(x) = \sin(\lambda_n x)$

TO USE AN EIGENFUNCTION EXPANSION LET: $-xt = \sum_{n=0}^{\infty} b_n(t) \sin\lambda_n x$

WHERE $b_n(t) = t\dfrac{2}{(\pi/2)} \displaystyle\int_0^{\pi/2} (-x\sin\lambda_n x)\,dx = \dfrac{4t}{\pi}\left[x\dfrac{\cos\lambda_n x}{\lambda_n}\Big|_0^{\pi/2} - \dfrac{1}{\lambda_n}\int_0^{\pi/2}\cos(\lambda_n x)\right]$

$$= \frac{4t}{\pi}\left[\cancelto{0}{\frac{(\pi/2)\cos((2n+1)\pi/2)}{\lambda_n}} - \frac{1}{\lambda_n^2}\sin((2n+1)x)\Big|_0^{\pi/2}\right]$$

$$= -\frac{4t}{\pi}\sin\left((2n+1)\frac{\pi}{2}\right) = C_n \cdot t \qquad C_n = -\frac{4}{\pi}\sin(\lambda_n \pi/2)/\lambda_n^2$$

NOW LET $v(x,t) = \sum_{n=0}^{\infty} \hat{v}_n(t)\sin(\lambda_n x)$; $v_t = \sum_{n=0}^{\infty} \dot{\hat{v}}_n \sin(\lambda_n x)$; $v_{xx} = \sum_{n=0}^{\infty} \hat{v}_n(-\lambda_n^2)\sin(\lambda_n x)$

$$0 = v_t - \alpha^2 v_{xx} + xt = \sum_{n=0}^{\infty}\left\{\dot{\hat{v}}_n(t) + \alpha^2\lambda_n^2\hat{v}_n - C_n t\right\}\sin(\lambda_n x)$$

$$\therefore \dot{\hat{v}}_n(t) + \alpha^2\lambda_n^2\hat{v}_n = C_n t \Rightarrow \left(e^{\alpha^2\lambda_n^2 t}\hat{v}_n\right)^{\cdot} = C_n t e^{\alpha^2\lambda_n^2 t}$$

$$e^{\alpha^2\lambda_n^2 t}\hat{v}_n = C_n\left[\frac{te^{\alpha^2\lambda_n^2 t}}{\alpha^2\lambda_n^2} - \frac{e^{\alpha^2\lambda_n^2 t}}{\alpha^4\lambda_n^4}\right] + d_n$$

$$\therefore \hat{v}_n(t) = C_n\left[\frac{t}{\alpha^2\lambda_n^2} - \frac{1}{\alpha^4\lambda_n^4}\right] + d_n e^{-\alpha^2\lambda_n^2 t}$$

$$\therefore v(x,t) = \sum_{n=0}^{\infty}\left\{C_n\left[\frac{t}{\alpha^2\lambda_n^2} - \frac{1}{\alpha^4\lambda_n^4}\right] + d_n e^{-\alpha^2\lambda_n^2 t}\right\}\sin(\lambda_n x)$$

$$\therefore 0 = v(x,0) = \sum_{n=0}^{\infty}\left[-\frac{C_n}{\alpha^4\lambda_n^4} + d_n\right]\sin(\lambda_n x)$$

$$\therefore \quad d_n = \frac{C_n}{\alpha^4\lambda_n^4}$$

$$\therefore u(x,t) = w(x,t) + v(x,t)$$

$$= \frac{t^2 x}{2} + \sum_{n=0}^{\infty} C_n\left\{\frac{t}{\alpha^2\lambda_n^2} - \frac{1}{\alpha^4\lambda_n^4}\left(1 - e^{-\alpha^2\lambda_n^2 t}\right)\right\}\sin\lambda_n x.$$

Where $\quad C_n = \dfrac{-4}{\pi}\,\dfrac{\sin\left((2n+1)\pi/2\right)}{\lambda_n^2} = \dfrac{4}{\pi}\,\dfrac{(-1)^{n+1}}{(2n+1)^2}$