Math 257/316, Midterm 1, Section 101/102
8 October 2010

Instructions. The exam lasts 55 minutes. Calculators are not allowed. A formula sheet is attached.

1. Consider the ODE

$$2x^2y'' + (x+\alpha)y' - (x+1)y = 0,$$

in which α is a constant.

(a) Classify the point $x = 0$ (as ordinary point, regular singular point, or irregular singular point) depending on the value of α. [5 marks]

(b) For $\alpha = 0$, find two independent solutions (for $x > 0$) in the form of series about $x = 0$ (you need only write the first three non-zero terms in each solution). [20 marks]

(a) the ODE is $y'' + \underbrace{\left[\frac{x+\alpha}{2x^2}\right]}_{:=p(x)} y' - \underbrace{\left[\frac{x+1}{2x^2}\right]}_{:=q(x)} y$

• both $p(x)$ and $q(x)$ become infinite as $x \to 0$
$\Rightarrow x=0$ is a singular point

• $\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{-(x+1)}{2} = -\frac{1}{2}$, finite

• $\lim_{x\to 0} x p(x) = \lim_{x\to 0} \left(\frac{1}{2} + \frac{\alpha}{2x}\right) = \begin{cases} \frac{1}{2} \text{ if } \alpha = 0 \\ \text{infinite, if } \alpha \neq 0 \end{cases}$ $\Rightarrow$

$\boxed{\begin{array}{l} x=0 \text{ is a} \\ \bullet \text{ regular singular point if } \alpha=0 \\ \bullet \text{ irregular singular point if } \alpha\neq 0 \end{array}}$

(b) $2x^2y'' + xy' - (x+1)y = 0$

• since $x=0$ is a R.S.P., try $y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$ $\Rightarrow$ $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$

$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$

$\Rightarrow$ we need

$$0 = 2x^2 \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - (x+1)\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+1}}_{n \mapsto n-1} - \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$" \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

$$= \left[2r(r-1) + r - 1\right]a_0 x^r + \sum_{n=1}^{\infty}\left(\left\{\left[2(n+r-1)+1\right](n+r) - 1\right\}a_n - a_{n-1}\right)x^{n+r}$$

• indicial equation: $0 = 2r(r-1)+r-1 = 2r^2 - r - 1 = (2r+1)(r-1) \Rightarrow$ roots $r_1 = 1$, $r_2 = -\frac{1}{2}$

• recurrence relation: $a_n = \dfrac{a_{n-1}}{[2(n+r)-1](n+r)-1}$; take $a_0 = 1$

• $\underline{r=1}$: $a_n = \dfrac{a_{n-1}}{(2n+1)(n+1)-1}$ $\Rightarrow$ $a_1 = \dfrac{1}{5}$, $a_2 = \dfrac{1/5}{14} = \dfrac{1}{5\cdot 14}$ $\Rightarrow$ $\boxed{y_1(x) = x\left[1 + \frac{1}{5}x + \frac{1}{5\cdot 14}x^2 + \dots\right]}$

$\underline{r=-\frac{1}{2}}$: $a_n = \dfrac{a_{n-1}}{(2n-2)(n-\frac{1}{2})-1}$ $\Rightarrow$ $a_1 = \dfrac{1}{-1} = -1$, $a_2 = \dfrac{-1}{2} = -\frac{1}{2}$ $\Rightarrow$ $\boxed{y_2(x) = x^{-1/2}\left[1 - x - \frac{1}{2}x^2 + \dots\right]}$

2. Consider the following heat equation problem with zero boundary conditions:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, \quad t > 0\\ \text{BC} &: u(0,t) = 0 = u(\pi,t)\\ \text{IC} &: u(x,0) = f(x)\end{aligned}$$

(a) Apply the method of separation of variables, and find the solution if $f(x) = 3\sin(2x) - \sin(4x)$. [15 marks]

(b) Find the Fourier series of the 2π-periodic function $f(x)$ with $f(x) = 3x$ on $-\pi \le x \le \pi$. [5 marks]
(*Hint:* $\int_{-\pi}^{\pi} x\sin(nx)dx = \frac{2\pi}{n}(-1)^{n+1} \qquad n = 1,2,3,\ldots$)

(c) Use your answer to (b) to find the solution of the above heat equation problem with $f(x) = 3x$. [5 marks]

(a) try $u(x,t) = X(x)T(t)$, so $XT' = X''T \Rightarrow \frac{T'}{T} = \frac{X''}{X} = \text{constant} = -\lambda$

X problem: $\left\{\begin{array}{l} X'' = -\lambda X \\ X(0) = X(\pi) = 0 \end{array}\right\}$ ← from (BC)

- if $\lambda > 0$, $X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$ (general soln.)
- $0 = X(0) = A \Rightarrow X(x) = B\sin(\sqrt{\lambda}x)$
- $0 = X(\pi) = B\sin(\sqrt{\lambda}\pi) \Rightarrow$ need $\sqrt{\lambda}\pi = n\pi$

or $\lambda = n^2 \quad n = 1,2,3,4,\ldots$

& $X(x) = (\text{const})\sin(nx)$

(• as in class, we cannot satisfy the boundary conditions if $\lambda \le 0$)

T problem: $T' = -\lambda T = -n^2 T \Rightarrow T(t) = (\text{const})e^{-n^2t}$

- thus we have a general solution $u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)e^{-n^2t}$ for (HE) and (BC)
- the (IC) is $f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx)$
- if $f(x) = 3\sin(2x) - \sin(4x)$, we can satisfy this with $\left\{\begin{array}{l} b_2 = 3,\ b_4 = -1 \\ \text{all other } b_n = 0 \end{array}\right\}$

$$\Rightarrow \boxed{u(x,t) = 3\sin(2x)e^{-4t} - \sin(4x)e^{-16t}}$$

(b) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$ with $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\cos(nx)(3x)}_{\text{odd}}dx = 0$ $n = 0,1,2,3\ldots$

integral over symmetric interval

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(nx)(3x)dx \overset{\text{using Hint}}{=} \frac{3}{\pi}\cdot\frac{2\pi}{n}(-1)^{n+1} = \frac{6}{n}(-1)^{n+1}$

$$\Rightarrow \boxed{f(x) = 6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)}$$

(c) since $f(x) = 3x$ for $-\pi \le x \le \pi$, we can satisfy (IC) above using $b_n = \frac{6}{n}(-1)^{n+1}$

$$\Rightarrow \boxed{u(x,t) = 6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)e^{-n^2t}}$$