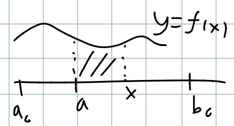
Lec 8 — F.T.C & applications §5.5.

Note TA's office hr. in MLC.

Thm (F.T.C.) 1. Suppose f continuous on $[a_0, b_0]$. $a \in [a_0, b_0]$.

Let $F(x) = \int_a^x f(t)\,dt$.



Then F is differentiable for $x \in (a_0, b_0)$

& $F'(x) = f(x)$

i.e. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

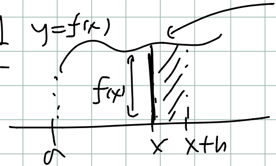
2. Suppose G is a differentiable function on $[a_0, b_0]$

& $G'(x)$ is bounded Riemann integrable on $[a_0, b_0]$.

Then $\int_a^b G'(x)\,dx = G(b) - G(a)$. $\forall\ a, b \in [a_0, b_0]$.

Huenstic ideas

Part 1



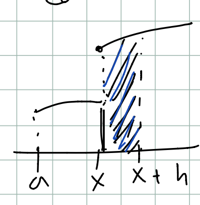
$F(x+h) - F(x)$

the rate of change of this area of

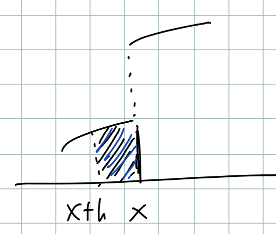
is the length of the boundary $f(x)$.

Q. Is continuity assumption on f essential in part 1. ?

e.g. Yes. Consider:



$h > 0$

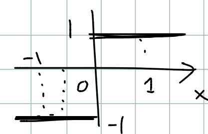


$h < 0$

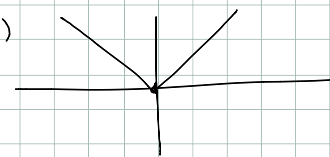
two different rate of changes of the area.

More concretely, consider.

e.g. $f(x) = \begin{cases} 1 & x<0 \\ -1 & x \geq 0 \end{cases}$



Let $F(x) = \int_0^x f(t)\,dt$:

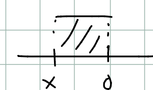


$F(0) = 0$

$x>0 \Rightarrow \quad F(x) = \int_0^x 1\,dt = x$

$x<0 \Rightarrow \quad F(x) = \int_0^x -1\,dt = -\int_0^x dt = -\left(-\int_x^0 dt\right)$

$= \int_x^0 dt = |x| = -x.$



$$F(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

$$\therefore \quad \frac{F(x)-F(0)}{x-0} = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$$

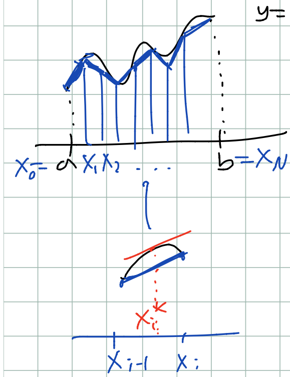
Thus $\lim_{x\to 0} \frac{F(x)-F(0)}{x-0}$ does NOT exist.

i.e. $F(x) = \int_0^x f(t)\,dt$ is not differentiable at $x=0$

By the same reason $F_2(x) = \int_{-1}^x f(t)\,dt$

is NOT differentiable at $x=0$. //

Hueristic explanation for part 2. (not a rigorous proof) (but, very close).



By mean-value thm,

$$G(x_i) = G(x_{i-1}) + G'(x_i^*)\,\Delta x_i$$

for some $x_i^* \in [x_{i-1}, x_i]$

telescopic sum

$$\Rightarrow \sum_{i=1}^{N} G'(x_i^*)\Delta x_i = G(x_N) - G(x_0).$$

$$\approx \int_a^b G'(t)\,dt$$

//

Proof of Part 2.

We skip the full proof, but
We prove only the special case where G' is continuous.

Assume G' is continuous. on $[a_0, b_0]$.

Fix $a, b \in [a_0, b_0]$

Then define $F(x) = \int_a^x G'(t)\,dt$.

From part 1, $F'(x) = G'(x)$. for $x \in [a_0, b_0]$.

Then from this & using Thm 13, Section 2.8.

$$F(x) = G(x) + C \quad \text{for some constant.}$$

Thus

$$\left.\begin{aligned} F(a) &= G(a) + C \\ F(b) &= G(b) + C \\ \text{but} \quad F(a) &= \int_a^a G'(t)\,dt = 0 \\ F(b) &= \int_a^b G'(t)\,dt \end{aligned}\right\} \Longrightarrow \underline{\int_a^b G'(t)\,dt = G(b) - G(a)}$$

$\boxtimes$

- Application of F.T.C. to solve differential equations

e.g. Find a differentiable function $F(x)$ such that $F'(x) = 2xe^{x^3}$ & $F(0)=0$.

(Sol) Let $F_1(x) = \int_0^x 2te^{t^3}\,dt$

continuous so can apply F.T.C. part 1.

Then $F_1(0) = 0$.

$$F_1'(x) = \frac{d}{dx}\int_0^x 2te^{t^3}\,dt = 2xe^{x^3} \text{ by F.T.C}$$

So $\underline{F_1(x) = \int_0^x 2te^{t^3}\,dt}$ ▨

- Applications of F.T.C. in evaluating definite integrals.

e.g. $\int_0^1 \sin x\,dx = -\cos(1) - (-\cos 0)$ notation $= \left[-\cos x\right]_{x=0}^{x=1}$

$(-\cos x)' = \sin x$

$= -\cos 1 + \cos 0$

$= \underline{-\cos 1 + 1}$.

e.g. $x>0$ $\int_1^x \frac{1}{t}\,dt = \ln t\Big]_1^x$ ← $(\ln x)' = \frac{1}{x}$ for $x>0$.

$= \ln x - \ln 1$

$= \underline{\ln x}$.

- Applications of F.T.C in solving integral equations.

e.g. Find a differentiable $f(x)$ such that

$$(*) \cdots \quad f(x) = 1 + \int_1^x f(t)\,dt$$

(sol) Try to find f among differentiable functions.

Assume a differentiable function f solves $(*)$.

Then F.T.C allows us differentiate $(*)$ (since f is continuous) and get

$$f'(x) = f(x) \leftarrow \frac{d}{dx}\int_1^x f(t)\,dt$$

$$\Rightarrow \quad f(x) = Ce^x.$$

Note $f(1) = 1 + \cancel{\int_1^1 f(t)\,dt}^{\,0} = 1.$

$\therefore 1 = f(1) = Ce^1 = Ce$

$\therefore C = e^{-1}$ //

$$\Rightarrow \quad f(x) = e^{-1}e^x = e^{x-1} \quad \boxed{/\!/}$$

<u>Rmk</u> Indeed, if we plug-in $f(x) = e^{x-1}$ in $(*)$

Then, LHS : e^{x-1}

RHS : $1 + \int_1^x e^{t-1}\,dt = 1 + e^{-1}\left[e^t\right]_1^x = 1 + e^{-1}\left[e^x - e^1\right]$

$= \cancel{1} + e^{x-1} - \cancel{1}$

So LHS = RHS ✓

- Such solution is unique.

Note If $g(x)$ was a solution to

$$g(x) = 1 + \int_1^x g(t)\,dt$$

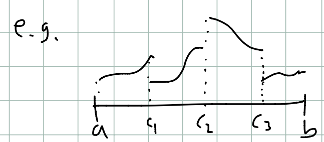
Then
- $g'(x) = g(x)$
- $g(1) = 1.$

$$\Rightarrow \quad g(x) = e^{x-1}. \quad \boxed{/\!/}$$

piecewise continuous functions on $[a,b]$

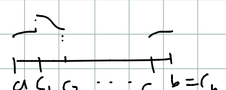
are the functions that are continuous
except at finitely many points.

e.g.



Thm Suppose f is a bounded piecewise continuous function on $[a,b]$

Then, f is integrable



Suppose further that f is given by for $a=c_0 < c_1 < c_2 \cdots < c_n = b$

$$f(x) = g_i(x) \quad \text{for } c_{i-1} \le x < c_i$$

g_i continuous on $[c_{i-1}, c_i]$.

Then $$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} g_i(x)\,dx$$

e.g. $$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 2 & 1 < x \le 3 \\ 3 & 3 < x \le 4 \end{cases}$$

$$\int_0^4 f(x)\,dx = \int_0^1 1\,dx + \int_1^3 2\,dx + \int_3^4 3\,dx$$

Comments about proof

The proof is optional.
You use integrability of continuous functions.
But, there are finitely many discontinuous points.

When you control $U(f,P) - L(f,P) = \sum_{i=1}^{N} (M_i - m_i)\Delta x_i$
Make the partition in such a way the contribution from discontinuity points are arbitrary small.
This proof has similar ideas (but more complicated) as WHW 1 Problem 3. (a).

$\boxed{/\!/}$