Lec 46. § 9.7 , Taylor Series.

Will Learn Today.

- Taylor series can be used to get approximate solutions to difficult problems.

  e.g.
  - $\int_0^1 e^{x^2}\,dx = ?$
  - Solve approximately.

    $$\frac{dy}{dx} = xy + 1$$

§. Applications of Taylor series.

- Approximatly computing integrals
- Approximately solving differential eqns.

EX. Find an approximate value of

$$\int_0^1 e^{x^3}\,dx \quad \text{using Taylor series.}$$

- <sol>.

- Idea:
  - Integrating e^{x^3} directly, seems impossible,

    many such examples e.g. e^{x^2}, e^{-x^2}, e^{x^4}, … $\cos(x^2)$, $\sin(x^2)$, …

    but, integrating its Taylor series expansion is easy.

$$\int_a^b \left[\sum_{n=0}^{\infty} a_n x^n\right] dx = \sum_{n=0}^{\infty} \underbrace{\int_a^b a_n x^n\,dx}_{\text{easy integral.}}$$

  - The integral of a Taylor series is a series.

  - Use this series, to approximate the value of the original integral.

- Note $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

- So, $e^{x^3} = \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}$ ← "simple substitution". replaced x with x^3.

$$= \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

- So, $\int_0^1 e^{x^3}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}\,dx$

$$= \sum_{n=0}^{\infty} \int_0^1 \frac{x^{3n}}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{n!} \frac{x^{3n+1}}{3n+1} \right]_0^1$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \cdot \frac{(1)^{3n+1}}{3n+1} - \cancel{\frac{1}{n} \cdot \frac{(0)^{3n+1}}{3n+1}} \right\} \quad \to 0$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!(3n+1)}$$

So, $\int_0^1 e^{x^3}\,dx \approx \frac{1}{0!(3\cdot 0+1)} + \frac{1}{1!(3\cdot 1+1)} + \frac{1}{2!(3\cdot 2+1)} + \frac{1}{3!(3\cdot 3+1)} + \frac{1}{4!(3\cdot 4+1)}$

approximation.

$$\underline{\int_0^1 e^{x^3}\,dx \approx 1 + \frac{1}{4} + \frac{1}{2\cdot 7} + \frac{1}{3\cdot 2\cdot 10} + \frac{1}{4\cdot 3\cdot 2\cdot 13}}$$

Ex Find $y(x)$ with $\frac{dy}{dx} = \sin(x^2)$, $y(0)=1$.

using Taylor series.

<sol> Basically we are computing $\int \sin(x^2)\,dx$

($\sin(x^2)$ does $\boxed{\text{not}}$ have a simple antiderivative.

Such examples are e^{x^2}, $\cos(x^2)$, $\cdots$)

- Power series of $\sin(x^2)$

Use $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$

So $\sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (x^2)^{2k+1}$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{4k+2}.$$

- $\int \sin(x^2)\,dx$ using the power series.

$$\int \sin(x^2)\,dx = \int \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{4k+2} \right] dx$$

Term by term integral

$$= \sum_{k=0}^{\infty} \int \frac{(-1)^k}{(2k+1)!} x^{4k+2}\,dx$$

additive constant for an indefinite integral.

$$= C + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{x^{4k+3}}{4k+3}$$

So, $y(x) = C + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{x^{4k+3}}{(4k+3)}$

with the constant C TO BE DETERMINED

- To determine C, use $y(0) = 1$. ← The GIVEN initial value.

$\therefore\ 1 = y(0) = C + \cancel{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{0^{4k+3}}{(4k+3)}}\ 0$

So, $\underline{C = 1}$

- Finally, $y(x) = 1 + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{x^{4k+3}}{(4k+3)}$ Answer.

remark

So, $y(x) = 1 + \frac{x^3}{3} - \frac{x^7}{6\cdot 7} + \frac{x^{11}}{5\cdot 4\cdot 3\cdot 2\cdot 11} - \cdots$

Approximately,

$y(x) \approx 1 + \frac{x^3}{3} - \frac{x^7}{6\cdot 7} + \frac{x^{11}}{5\cdot 4\cdot 3\cdot 2\cdot 11}$ ▨

approximation that works well for x close to 0 (i.e. small $|x|$).

Ex. Solve $\frac{dy}{dx} = xy + 1$, & $y(0) = 1$.

using Taylor series.

(Determine up to degree 7 terms)

<sol>. • Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$.

Want: y(x) solve the diff. eqn.

① Plug-in y(x) into the differential eqn.

② will get conditions on c_n.

③ Use these conditions to determine c_n.

①: Plug-in $y(x) = \sum_{n=0}^{\infty} c_n x^n$

into $\frac{dy}{dx} = xy + 1$.

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n x^n\right] = x\left[\sum_{n=0}^{\infty} c_n x^n\right] + 1.$$

careful.

$$\sum_{n=0}^{\infty}\left[\frac{d}{dx} c_n x^n\right] = \sum_{n=0}^{\infty} c_n x^{n+1} + 1.$$

$$\Rightarrow \sum_{n=1}^{\infty} c_n \cdot n\, x^{n-1} = \sum_{n=0}^{\infty} c_n x^{n+1} + 1.$$

NOTICE

② To get conditions on c_n.

Compare the same degree terms.

- To do this, rearrange the summation

$$\sum_{n=1}^{\infty} c_n \cdot n\, x^{n-1} = \sum_{n=0}^{\infty} c_n x^{n+1} + 1.$$

the SAME degrees.

to

$$\underbrace{\sum_{k=0}^{\infty} c_{k+1}(k+1)\, x^{k}}_{\substack{k=n-1 \\ n=k+1.}} = \sum_{k=1}^{\infty} \underbrace{c_{k-1} x^{k}}_{\substack{k=n+1 \\ n=k-1}} + \underbrace{1}_{\substack{\text{0-th degree term} \\ (k=0).}}$$

Look at the k-th degree terms on both side.

(with the same k).

See :

- $c_{k+1}(k+1) = c_{k-1}$ for $k \geq 1$.
- $c_{0+1}(0+1) = 1$ for $k=0$

So,

$$\boxed{\begin{cases} c_{k+1} = \dfrac{c_{k-1}}{k+1} & \text{for } k \geq 1 \\ c_1 = 1. \end{cases}}$$

$C_0 = ?$: Note: $y(x) = \sum_{n=0}^{\infty} C_n x^n$

$= C_0 + C_1 x + C_2 x^2 + \cdots$

So, $y(0) = C_0$.

From initial value : $y(0) = 1$.

the given

So, $C_0 = 1$

③. Determine C_n's.

From ②, we have

From $y' = xy + 1$

- $C_{k+1} = \dfrac{C_{k-1}}{k+1}$ for $k \geq 1$
- $C_1 = 1$

From $y(0) = 1$. →

- $C_0 = 1$

Use the recursive relation

$C_{k+1} = \dfrac{C_{k-1}}{k+1}$. : $C_{k-1} \longrightarrow C_k$

two steps.

$\bullet\ C_0 = 1, \quad C_2 = \frac{C_0}{2} = \frac{1}{2}. \quad C_4 = \frac{C_2}{4} = \frac{1}{4\cdot 2}$

$$C_6 = \frac{C_4}{6} = \frac{1}{6\cdot 4\cdot 2}.$$

<u>Pattern</u>: $C_{2k} = \frac{1}{(2k)\cdot(2k-2)\cdot \cdots \cdot 4\cdot 2}$ for $k \geq 1$.

$$= \frac{1}{2\cdot k\cdot 2\cdot(k-1)\cdot 2(k-2)\cdots 2(2)\cdot 2\cdot 1}$$

$$= \frac{1}{2^k\cdot k\cdot(k-1)\cdot\cdots\cdot 3\cdot 2\cdot 1}$$

$$= \frac{1}{2^k\, k!}.$$

$\bullet\ C_1 = 1. \quad C_3 = \frac{C_1}{3} = \frac{1}{3}. \quad C_5 = \frac{C_3}{5} = \frac{1}{5\cdot 3}$

$$C_7 = \frac{C_5}{7} = \frac{1}{7\cdot 5\cdot 3}$$

$\vdots$

<u>Pattern</u>: $C_{2k+1} = \frac{1}{(2k+1)(2k-1)\cdots 5\cdot 3\cdot 1}$ for $k \geq 1$

this denominator is NOT a factorial.

Finally, put together.

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + \cdots$$

$$= 1 + x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4 \cdot 2} x^4 + \frac{1}{5 \cdot 3} x^5 + \frac{1}{6 \cdot 4 \cdot 2} x^6 + \frac{1}{7 \cdot 5 \cdot 3} x^7 + \cdots$$

$$= \boxed{1 + x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{8} x^4 + \frac{1}{15} x^5 + \frac{1}{48} x^6 + \frac{1}{105} x^7 + \cdots}$$

▨