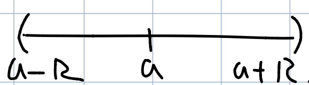
Lec 44

Let us prove:

Taylor's thm

- **Suppose:** $f(x)$ is differentiable up to $(k+1)$-times on an interval $(a-R, a+R)$



Let

- $T_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k$

  Taylor polynomial of degree k, centered at $x=a$.

- $R_k(x) = f(x) - T_k(x)$

  $R_k(x)$ is the error of approximation to $f(x)$ by the Taylor polynomial

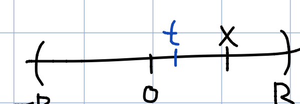
THEN for $a-R < x < a+R$

$$R_k(x) = \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\,dt.$$

proof We show this by repeating the fundamental theorem of calculus.

For simplicity, let $a=0$ without loss of generality (since can consider $h(x) = f(x+a)$)

- $f(x) = f(0) + \int_0^x f'(t)\,dt.$



$f'(x) = f'(0) + \int_0^x f''(t)\,dt$

$\vdots$

$f^{(k)}(x) = f^{(k)}(0) + \int_0^x f^{(k+1)}(t)\,dt$

• Try to find the pattern:

$$f(x) = f(0) + \int_0^x \left[f'(0) + \int_0^{t_1} f''(t_2)\,dt_2\right] dt_1$$

$$= f(0) + f'(0)x \quad + \int_0^x \left[\int_0^{t_1} f''(t_2)\,dt_2\right] dt_1$$

$$= f(0) + f'(0)x \quad + \int_0^x \int_0^{t_1} \left[f''(0) + \int_a^{t_2} f'''(t_3)\,dt_3\right] dt_2\,dt_1$$

$$= f(0) + f'(0)x \quad + \int_a^x \left[\int_a^{t_1} f''(0)\,dt_2\right] dx$$

$$+ \int_0^x \left[\int_0^{t_1} \left[\int_0^{t_2} f'''(t_3)\,dt_3\right] dt_2\right] dt_1$$

$$= f(0) + f'(0)x \quad + f''(0) \int_0^x \left[\int_0^{t_1} dt_2\right] dt_1$$

$$+ \int_0^x \int_0^{t_1} \int_0^{t_2} \left[f'''(0) + \int^{t_3} f(t_4)\,dt_4\right] dt_3\,dt_2\,dt_1$$

$$= f(\ ) + f'(0)x + f''(0) \int_0^x \left[\int_0^{t_1} dt_2\right] dt_1$$

$$+ f'''(0) \int_0^x \int_0^{t_1} \int_0^{t_2} dt_3\,dt_2\,dt_1$$

$$+ \int_0^x \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} f^{(4)}(t_4)\,dt_4\,dt_3\,dt_2\,dt_1$$

In this way,
we see:

$$f(x) = f(\ ) + f'(0) \quad + f''(0) \int_0^x \int_0^{t_1} dt_2\,dt_1 + f'''(0) \int_0^x \int_0^{t_1} \int_0^{t_2} dt_3\,dt_2\,dt_1$$

$$+ \cdots + f^{(k)}(0) \int_0^x \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{k-1}} dt_k\,dt_{k-1} \cdots dt_2\,dt_1$$

$$+ \int_0^x \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_k} f^{(k+1)}(t_{k+1})\,dt_{k+1}\,dt_k \cdots dt_2\,dt_1 .$$

<u>Now</u> the k-th term is

↙ name.

$$f^{(k)}(0) \int_0^x \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{k-1}} dt_k\,dt_{k-1} \cdots dt_2\,dt_1 = f^{(k)}(0)\, I_k$$

and

$$R_k(x) = \int_0^x \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_k} f^{(k+1)}(t_{k+1})\,dt_{k+1}\,dt_k \cdots dt_2\,dt_1 .$$

Compute

$$I_k = \int_0^x \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{k-1}} dt_k \, dt_{k-1} \cdots dt_2 \, dt_1$$

$$= \int_0^x \int_0^{t_1} \cdots \int_0^{t_{k-2}} t_{k-1} \, dt_{k-1} \cdots dt_2 \, dt_1$$

$$= \int_0^x \int_0^{t_1} \cdots \int_0^{t_{k-3}} \frac{t_{k-2}^2}{2} \, dt_{k-2} \cdots dt_2 \, dt_1$$

$$= \int_0^x \int_0^{t_1} \cdots \int_0^{t_{k-4}} \frac{t_{k-3}^3}{2 \cdot 3} \, dt_{k-3} \cdots dt_2 \, dt_1$$

$$\cdots$$

$$= \int_0^x \frac{t_1^{k-1}}{2 \cdot 3 \cdots (k-1)} \, dt_1 = \frac{x^k}{k!}$$

At this moment, we showed

$$f(x) - T_k(x) = R_k(x) = \int_0^x \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_k} f^{(k+1)}(t_{k+1}) \, dt_{k+1} \, dt_k \cdots dt_2 \, dt_1 .$$

At this moment, if

we have $\left| f^{(k+1)}(s) \right| \le M \quad \forall \quad |s| < R$

then, $\left| R_k(x) \right| = \left| \int_0^x \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_k} f^{(k+1)}(t_{k+1}) \, dt_{k+1} \cdots dt_2 \, dt_1 \right|$

$$\le \int_0^{|x|} \left| \int_0^{t_1} \cdots \int_0^{t_k} M \, dt_{k+1} \, dt_k \cdots dt_2 \right| dt_1$$

$$\le \int_0^{|x|} \int_0^{|t_1|} \left| \quad \cdots \quad \right| dt_2 \, dt_1$$

$$\vdots$$

$$\le \int_0^{|x|} \int_0^{|t_1|} \int_0^{|t_2|} \cdots \int_0^{|t_k|} M \, dt_{k+1} \, dt_k \cdots dt_1$$

$$= M \cdot \int_0^{|x|} \cdots \int_0^{|t_{k-1}|} |t_k| \, dt_k \cdots dt_1$$

$$= M \int_0^{|x|} \cdots \int_0^{|t_{k-2}|} \frac{1}{2} |t_{k-1}|^2 \, dt_{k-1} \cdots dt_1$$

$$= \frac{M}{(k+1)!} |x|^{k+1} .$$ which is a desired estimate.

To compute $R_k(x)$, we do <u>not</u> want to

compute the iterated integral

$$\int_0^x \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_k} f^{(k+1)}(t_{k+1})\, dt_{k+1}\, dt_k \cdots dt_2\, dt_1 .$$

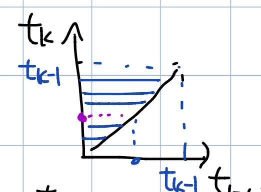
as in the given order,

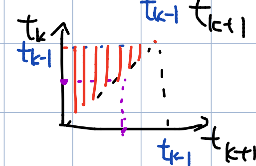
since if we do we will just get $f(x) - T_k(x)$,

without gaining any further information.

We try to reorder the iterated integral.

Reordering:





$$\int_0^{t_{k-1}} \int_0^{t_k} f^{(k+1)}(t_{k+1})\, dt_{k+1}\, dt_k$$

$$= \int_0^{t_{k-1}} \int_{t_{k+1}}^{t_{k-1}} f^{(k+1)}(t_{k+1})\, dt_k\, dt_{k+1}$$

$$= \int_0^{t_{k-1}} f^{(k+1)}(t_{k+1}) \left[\int_{t_{k+1}}^{t_{k-1}} dt_k \right] dt_{k+1}$$

$$= \int_0^{t_{k-1}} f^{(k+1)}(t_{k+1}) (t_{k-1} - t_{k+1})\, dt_{k+1}$$

So, $$\int_0^{t_{k-2}} \int_0^{t_{k-1}} \int_0^{t_k} f^{(k+1)}(t_{k+1})\, dt_{k+1}\, dt_k\, dt_{k-1}$$

$$= \int_0^{t_{k-2}} \int_0^{t_{k-1}} f^{(k+1)}(t_{k+1})(t_{k-1} - t_{k+1})\, dt_{k+1}\, dt_{k-1}$$

by the same principle

$$= \int_0^{t_{k-2}} \int_{t_{k+1}}^{t_{k-2}} f^{(k+1)}(t_{k+1})(t_{k-1} - t_{k+1})\, dt_{k-1}\, dt_{k+1}$$

$$= \int_0^{t_{k-2}} f^{(k+1)}(t_{k+1}) \frac{(t_{k-2} - t_{k+1})^2}{2}\, dt_{k+1}$$

$$\int_0^{t_{k-3}} \int_0^{t_{k-2}} \int_0^{t_{k-1}} \int_0^{t_k} f^{(k+1)}(t_{k+1})\, dt_{k+1}\, dt_k\, dt_{k-1}\, dt_{k-2}$$

$$= \int_0^{t_{k-3}} f^{(k+1)}(t_{k+1}) \frac{(t_{k-3} - t_{k+1})^3}{2 \cdot 3}\, dt_{k+1}$$

$$\cdots$$

$$R_k(x) = \int_0^x \int_0^{t_1} f^{(k+1)}(t_{k+1}) \frac{(t_1 - t_{k+1})^{k-1}}{1 \cdot 2 \cdot \cdots \cdot (k-1)}\, dt_{k+1}\, dt_1$$

$$= \int_0^x \int_{t_1}^x f^{(k+1)}(t_{k+1}) \frac{(t_1 - t_{k+1})^{k-1}}{(k-1)!}\, dt_1\, dt_{k+1}$$

$$= \int_0^x f^{(k+1)}(t_{k+1}) \frac{(x - t_{k+1})^k}{k!}\, dt_{k+1}$$

$$= \int_0^x f^{(k+1)}(t) \frac{(x-t)^k}{k!}\, dt.$$

$\blacksquare$