Lec 42. Differentiation / Integration of Power series : proof. § 9.5

Differentiation & Integration.

Thm Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ $\quad -R < x < R$

(i.e. the series converges on $(-R, R)$)
(So, absolutely convergent for each x, $|x| < R$).

Then, A. $f(x)$ is differentiable on $(-R, R)$

term-by-term differentiation: & $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$ on $(-R, R)$

B. $\forall |x| < R$,

term-by-term Integration: $\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n\,dt$

$= \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ on $(-R, R)$.

$= a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots$ for $-R < x < R$.

both series converge absolutely on $(-R, R)$

※ Note: the radius of convergence is the same before / after differentiation / integration.
- Interval of convergence may change.

Proof of thm

① Assume A. show B.

② show A.

① : Assume A. & $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$. (Then it absolutely converges for $|x| < R$ by the previous theorem.)

Consider $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

It converges absolutely for $|x| < R$ since

limit comparison: $\left| \frac{\frac{a_n x^{n+1}}{n+1}}{a_n x^n} \right| = \frac{|x|}{n+1} \to 0 < 1$.

So, from A, it is differentiable for $|x| < R$

& $\left(\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \right)' = \sum_{n=0}^{\infty} a_n x^n$

This shows B, since $h(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ $\quad |x| < R$

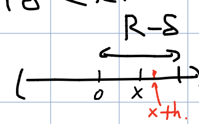
satisfies $h(0) = 0$
$h'(x) = \sum_{n=0}^{\infty} a_n x^n$

i.e. $h(x) = \int_0^x \left[\sum_{n=0}^{\infty} a_n t^n \right] dt$.

② : Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ which converges absolutely for $|x| < R$.

We will prove ② for $|x| < R$.

Fix x, $|x| < R$. Choose $\delta > 0$ s.t. $|x| + \delta < R$.



- For small $0 < |h| < \delta$.

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left[\sum_{n=0}^{\infty} a_n (x+h)^n - \sum_{n=0}^{\infty} a_n x^n\right] \quad \text{for } \cdot |x+h| < R, \ \cdot |x| < R$$

Note: This series is convergent absolutely for $\cdot |x+h| < R$, $\cdot |x| < R$.

$$= \frac{1}{h} \sum_{n=0}^{\infty} a_n \left[(x+h)^n - x^n\right]$$

$$= \frac{1}{\cancel{h}} \sum_{n=0}^{\infty} a_n \, n \, (c_n)^{n-1} \cancel{h} \quad \text{for some } c_n \text{ between } x \ \& \ x+h.$$

note: Mean value therem

$f(x+h) - f(x)$

$= f'(c) \cdot h$

for some $c \in [x, x+h]$ (or $c \in [x+h, x]$ if $h < 0$).

$\therefore (x+h)^n - x^n$

$= n (c_n)^{n-1} h \qquad x \le c_n \le x+h$

$$= \sum_{n=0}^{\infty} a_n \, n \, (c_n)^{n-1}.$$

- Now, want to take $h \to 0$.

Left side $= f'(x)$.

Right side $= \lim_{h \to 0} \left[\sum_{n=0}^{\infty} a_n \, n \, (c_n)^{n-1}\right]$

If the limits exist.

Can we do $\lim_{h\to 0} \left[\sum_{n=0}^{\infty} a_n \, n \, (c_n)^{n-1}\right] = \sum_{n=0}^{\infty} \lim_{h \to 0} a_n \, n \, (c_n)^{n-1}$??

In particular, does such limit on the left hand side exist?

(If so, note $\lim_{h\to 0} c_n = x$ since $c_n \in [x, x+h]$ or $[x+h, x]$.)

Note In $\lim_{x \to c} \sum_{n=0}^{\infty} b_n x^n$, the series $\sum_{n=0}^{\infty} \left[b_n \lim_{x \to c} x^n \right]$ may NOT converge.

e.g. $\sum_{n=1}^{\infty} \frac{r^n}{n}$ as $r \to 1$

Step 1 $\sum_{n=0}^{\infty} |a_n \cdot n\, x^{n-1}|$ converges for each $|x| < R$

(°o°)

Fix $|x|<R$, let $|x|+\varepsilon<R$. So, the series $\sum_{n=1}^{\infty} a_n (|x|+\varepsilon)^n < \infty$

- Now, observe $n|x|^{n-1} \le \frac{1}{\varepsilon}(|x|+\varepsilon)^n$ (Since $(a+b)^n = a^n + n a^{n-1} b + \dots + b^n$.)

Thus $\sum_{n=1}^{\infty} |a_n| n |x|^{n-1} \le \sum_{n=1}^{\infty} \frac{1}{\varepsilon} |a_n| (|x|+\varepsilon)^n < \infty$.

This shows absolute convergence of $\sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$ for any $|x|<R$.

$\boxtimes$

Step 2 $\left| \sum_{n=1}^{\infty} n a_n c_n^{n-1} - \sum_{n=1}^{\infty} n a_n x^{n-1} \right| \to 0$ as $h \to 0$.

This implies that

$$\frac{f(x+h)-f(x)}{h} = \sum_{n=1}^{\infty} n a_n c_n^{n-1} \xrightarrow[h \to 0]{} \sum_{n=1}^{\infty} n a_n x^{n-1}.$$ showing ②.

(°o°)

- $$\left| \sum_{n=1}^{\infty} n a_n \left[c_n^{n-1} - x^{n-1} \right] \right|$$

$$\le \sum_{n=1}^{\infty} \left| n a_n \right| c_n^{n-1} - x^{n-1} \big] \big|$$

$$\le \sum_{n=1}^{\infty} \left| n a_n (n-1) [\tilde{c}_n]^{n-2} \cdot h \right|$$

$$= h \sum_{n=1}^{\infty} \left| n(n-1) a_n [\tilde{c}_n]^{n-2} \right|$$

By M.V.T. for some $\tilde{c}_n$ $(x \le \tilde{c}_n \le x+h)$

$|c_n^{n-1} - x^{n-1}|$

$= \left| (n-1) [\tilde{c}_n]^{n-2} (x - c_n) \right|$

$\le (n-1) [\tilde{c}_n]^{n-2} h$

- Note for small $h > 0$, in particular, $|h| < \frac{\delta}{2}$.

$$\left|(n-1)[\tilde{c}_n]^{n-2}\right| \leq \frac{[|\tilde{c}_n| + \frac{\delta}{2}]^{n-1}}{\delta/2}$$

using $(n-1)a^{n-2} \cdot b \leq (a+b)^{n-1}$

$$\leq \frac{(|x| + h + \frac{\delta}{2})^{n-1}}{\frac{\delta}{2}}$$

$$\leq \frac{(|x| + \delta)^{n-1}}{\delta/2}$$

- So, $\left| h \sum_{n=1}^{\infty} n(n-1) a_n [\tilde{c}_n]^{n-2} \right|$

$$\leq h \sum_{n=1}^{\infty} \frac{1}{\delta/2} \, n \, a_n (|x| + \delta)^{n-1}$$

$$= \frac{2h}{\delta} \underbrace{\sum_{n=1}^{\infty} n \, a_n (|x| + \delta)^{n-1}}_{\text{this series is convergent from Step 1.}} \quad \therefore < +\infty$$

as $h \to 0$

$$\longrightarrow 0. \quad \blacksquare$$