Lec 41. Power Series §9.5.

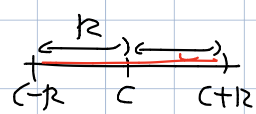
"operations on power series."

Given a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$

the set where it converges are of the form: either

$(c-R, c+R)$
$[c-R, c+R]$
$(c-R, c+R]$
$[c-R, c+R)$.
$(-\infty, \infty)$

← such set is called **the interval of convergence.**



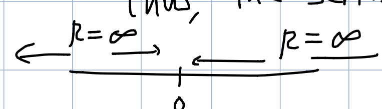
boundary points may or may not be included.

Example Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(sol). Ratio test:

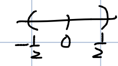
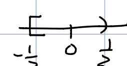
$$\left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \left| \frac{x}{n+1} \right| \to 0 \text{ as } n \to \infty \quad \forall x.$$

Thus, the series converges $\forall x$.



interval of convergence $R = +\infty$. ∎

e.g. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges for all x.

- $\sum_{n=1}^{\infty} n!\, x^n$ converges only at $x=0$.
- $\sum_{n=1}^{\infty} 2^n x^n$ converges exactly when $-\frac{1}{2} < x < \frac{1}{2}$ 
- $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$ converges exactly when $-\frac{1}{2} \leq x < \frac{1}{2}$ 

- <u>Properties of power series</u>

  <u>and how we can manipulate them.</u>

  (sum, product, differentiation, integration, ..)

- "Operations on power series are performed <u>TERM BY TERM</u>."

- "Operations on power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ are performed on the "open" interval

$$(a-R,\ a+R)$$

R = radius of convergence.

(where the series converges absolutely).

the same center

- Sum: $\sum_{n=0}^{\infty} c_n (x-a)^n + \sum_{n=0}^{\infty} d_n (x-a)^n$

$$= \sum_{n=0}^{\infty} (c_n + d_n)(x-a)^n$$

← addition TERM BY TERM.

Combine terms with the same degree.

the same center a.

Operations on Power series

- $c \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} c a_n x^n$ for those x such that the series $\sum_{n=0}^{\infty} a_n x^n$ converges.

- $\sum_{n=0}^{\infty} (a_n + b_n) x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n$ for those x such that both of the series $\sum_{n=0}^{\infty} a_n x^n$, $\sum_{n=0}^{\infty} b_n x^n$ converges.

e.g. $p(x) = 1 + x + x^2 + x^3 + x^4 + \cdots$

$= \sum_{n=0}^{\infty} x^n$ ($R = 1$). radius of convergence

$q(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$

$= \sum_{n=0}^{\infty} (n+1) x^n$ ($R=1$).

$p(x) + q(x) = 2 + 3x + 4x^2 + 5x^3 + \cdots$

$= \sum_{n=0}^{\infty} (n+2) x^n$ ($R=1$).

Product (useful!).

e.g. $(2+3x)(1+x+x^2+x^3+x^4+\cdots)$

$= 2 \cdot (1+x+x^2+x^3+\cdots)$
$+ 3x(1+x+x^2+x^3+\cdots)$ ← distribution $(a(x)+b(x))p(x) = a(x)p(x) + b(x)p(x)$.

$= (2 + 2x + 2x^2 + 2x^3 + \cdots)$
$+ (3x^{0+1} + 3x^{1+1} + 3x^{2+1} + \cdots)$ ← term by term products.

$= 2 + (2+3)x + (2+3)x^2 + (2+3)x^3 + \cdots$ ← term by term summation.

• Power series as functions.

Differentiation & Integration.

Thm Let $f(x) = \sum_{n=0}^{\infty} a_n x^n \quad -R < x < R$

(i.e. the series converges on $(-R, R)$)

(So, absolutely convergent for each x, $|x| < R$).

Then, A. $f(x)$ is differentiable on $(-R, R)$

term-by-term differentiation: & $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$ on $(-R, R)$

B. $\forall\, |x| < R$,

term-by-term integration: $\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n\,dt$

$= \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ on $(-R, R)$.

$= a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots$ for $-R < x < R$.

both series converge absolutely on $(-R, R)$

※ Note: the radius of convergence is the same before/after differentiation/integration.

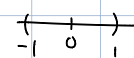
- Interval of convergence may change.

e.g. $\sum_{n=1}^{\infty} \frac{x^n}{n}$ interval of convergence: $-1 \le x < 1$

differentiation: $\sum_{n=1}^{\infty} \frac{n}{n} x^{n-1} = \sum_{k=0}^{\infty} x^k$: Interval of convergence $-1 < x < 1$. □

※ These properties are useful to find power series expression (Taylor series) of a function from known examples.

e.g.



- We know: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (valid for $|x|<1$).

So, by the properties of power series,

- $$\underbrace{\left(\frac{1}{1-x}\right)'}_{\text{the derivative is } \frac{1}{(1-x)^2} \text{ by direct computation}} = \sum_{n=1}^{\infty} n x^{n-1} \quad (\text{valid for } |x|<1)$$

on the left hand side.

Comparing the left & right hand sides, we conclude

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad (\text{valid for } |x|<1)$$

So, we obtained the power series expression for the function $\frac{1}{(1-x)^2}$

- $$\underbrace{\int \frac{1}{1-x}\,dx}_{\text{direct computation: } -\ln|1-x|} = \underbrace{\int \sum_{n=0}^{\infty} x^n \, dx}_{\text{term by term integration}} \quad (\text{valid for } |x|<1)$$

$$= \sum_{n=0}^{\infty} \int x^n \, dx$$

$$= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

So comparing both sides, we conclude that

$$-\ln|1-x| = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}. \quad \text{for some } C.$$

(valid for $|x|<1$)

To determine C, put $x=0$.

$$-\ln|1-0| = C + 0 \quad \therefore C=0.$$

Therefore,

$$-\ln|1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad \left(\text{valid for } |x|<1\right)$$

i.e. $$\ln|1-x| = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}. \quad (\text{valid for } |x|<1).$$

- Simple substitutions

can put a simple function cx, $x+b$, cx^k etc into a power series.

- e.g. From $\ln|1-x| = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ for $|x|<1$.

We see

$$\ln|1+x| = \ln|1-(-x)| = -\sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1} \text{ for } |-x|<1.$$

i.e.

$$\underbrace{\ln(1+x)}_{} = \sum_{n=0}^{\infty} \frac{-(-1)^{n+1} x^{n+1}}{n+1} \qquad \text{for } |x|<1$$

$1+x>0$ for $|x|<1$.

i.e.

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \quad \text{for } |x|<1}$$

equivalently,

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} \quad \text{for } |x|<1$$

e.g. From $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots$

$$= \sum_{n=0}^{\infty} x^n \quad \text{for } |x|<1$$

get $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1+(-x)+(-x)^2+(-x)^3+\cdots$

$$= \sum_{n=0}^{\infty} (-x)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{for } |x|<1$$

e.g.

From $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ ← note $0!=1$, $1!=1$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

(We will see later (Lec 35) why this is so.)

We get

$$e^{x^3} = 1 + x^3 + \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3!} + \frac{(x^3)^4}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}.$$

EX Use power series to approximate

$$\int_0^{\frac{1}{2}} \frac{dt}{1+t^3}$$

<sol>. Note that it is very difficult to find $\int \frac{dt}{1+t^3}$

- From the Power series

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots = \sum_{n=0}^{\infty} x^n$$

get (with simple substitution $x=-t^3$)

$$\frac{1}{1+t^3} = \frac{1}{1-(-t^3)} = 1+(-t^3)+(-t^3)^2+(-t^3)^3+\cdots$$

$$= \sum_{n=0}^{\infty} (-t^3)^n = \sum_{n=0}^{\infty} (-1)^n t^{3n}.$$ valid for $|t^3|<1$

i.e. for $|t|<1$

i.e. for $-1<t<1$.

- Do the integration of the power series (term by term)

$$\int_0^{\frac{1}{2}} \frac{dt}{1+t} = \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n t^{3n}\, dt$$

$$= \int_0^{\frac{1}{2}} 1\, dt + \int_0^{\frac{1}{2}} -t^3\, dt + \int_0^{\frac{1}{2}} t^6\, dt + \cdots$$

$$= \sum_{n=0}^{\infty} \int_0^{\frac{1}{2}} (-1)^n t^{3n}\, dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \left[\frac{t^{3n+1}}{3n+1}\right]_0^{1/2}$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{\left(\frac{1}{2}\right)^{3n+1}}{3n+1}$$

$$= \underbrace{\frac{1}{2}}_{n=0} - \underbrace{\frac{\left(\frac{1}{2}\right)^4}{4}}_{n=1} + \underbrace{\frac{\left(\frac{1}{2}\right)^7}{7}}_{n=2} - \underbrace{\frac{\left(\frac{1}{2}\right)^{10}}{10}}_{n=3.} + \cdots$$

$$\approx 0.48540144.$$ ▨