# Lec 40 . Convergence of power series, radius of convergence

§9.5

e.g. $\sum_{j=1}^{\infty} a_j$, $a_j = \begin{cases} \frac{1}{j} & j \text{ odd} \\ \frac{-1}{j^2} & j \text{ even} \end{cases}$

abs. converge? conditionally converge?

<sol.> $\sum_{j=1}^{\infty} |a_j| = 1 + \frac{1}{2^2} + \frac{1}{3} + \frac{1}{4^2} + \cdots$

$$\sum_{j=1}^{2N} |a_j| = \underbrace{1 + \frac{1}{3} + \cdots + \frac{1}{2N-1}}_{\to \infty \text{ as } N \to \infty} + \frac{1}{2^2} + \frac{1}{4^2} + \cdots + \frac{1}{(2n)^2}$$

∴ does NOT absolutely converge.

$$\sum_{j=1}^{2N} a_j = \underbrace{1 + \frac{1}{3} + \cdots + \frac{1}{2N-1}}_{\substack{\to \infty \\ \text{as } N \to \infty}} - \Big( \underbrace{\frac{1}{2^2} + \frac{1}{4^2} + \cdots + \frac{1}{(2n)^2}}_{\substack{\text{converges} \\ \text{as } N \to \infty}} \Big)$$

$$\xrightarrow[N \to \infty]{} +\infty .$$

∴ Does not conditionally converge. ▨

This is an example where conditions for alternating series do not hold.

· Convergence of power series, §9.5

radius of convergence

EX Find all x such that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} x^n \quad \text{converges.}$$

<sol>. $a_n = \frac{1}{\sqrt{n}} x^n$

Ratio test : $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{\sqrt{n+1}} x^{n+1}}{\frac{1}{\sqrt{n}} x^n}\right| = \sqrt{\frac{n}{n+1}}|x| \longrightarrow |x|$ as $n \to \infty$.

∴ The series converges absolutely for $|x| < 1$.

diverges for $|x| > 1$

· $x = 1$ : $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges to $+\infty$ by integral test $\int_1^{\infty} \frac{1}{\sqrt{x}} dx = +\infty$

· $x = -1$ : $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.. Alternating series & $\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}} \geq \frac{1}{\sqrt{n+2}}$, $\xrightarrow[\text{as } n\to\infty]{} 0$.

Thus, converges.

$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ converges exactly when $\underline{-1 \leq x < 1}$ ▨

## Power series: § 9.5

$$\sum_{n=0}^{\infty} \underbrace{a_n}_{\text{coefficient}} (x-\underbrace{c}_{\text{center}})^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

**Thm** For $\sum_{n=0}^{\infty} a_n (x-c)^n$, the three alternatives (only one of them holds)

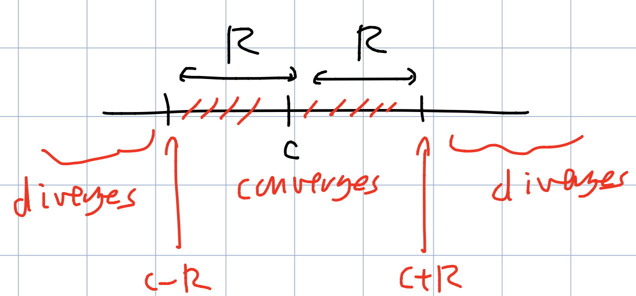
(i) it converges only at $x=c$ (& diverges if $x \neq c$)

(ii) it converges for all $x$

(iii) $\exists\, R > 0$ such that
it converges absolutely for $|x-c| < R$
& diverges for $|x-c| > R$

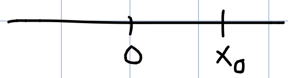
Called "Radius of convergence".

( Anything can happen for $x = c \pm R$ ).




it may or may not converge there.

(proof of theorem) Without loss of generality, can let $c=0$.




It suffices to show:

Claim If $\sum_{n=0}^{\infty} a_n x_0^n$ converges, then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely whenever $|x_0|<|x|$.

$\therefore$ Assume · $\sum_{n=1}^{\infty} a_n x_0^n$ converges

· $|x|<|x_0|$. i.e. $\frac{|x|}{|x_0|}<1$.

Since $|a_n x_0^n| \xrightarrow{n\to\infty} 0$, $\exists N$ s.t. $|a_n x_0|^n < 1$ $\forall n \geq N$.

Then
$$\sum_{n=N}^{\infty} |a_n x^n| = \sum_{n=N}^{\infty} |a_n x_0^n| \left|\frac{x}{x_0}\right|^n$$
$$\leq \sum_{n\geq N}^{\infty} \left|\frac{x}{x_0}\right|^n = \frac{1}{1-\left|\frac{x}{x_0}\right|} < \infty.$$

$\uparrow$ $\left|\frac{x}{x_0}\right|<1$

Thus, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. $\boxtimes$