Lec 38. "more convergence tests for series".

· limit comparism test
· ratio test
· root test.

§9.3.

Limit comparison test.

In the comparism test, what matters is the behavior of the terms of a given series as n gets larger.

Thus, we have:

Limit comparison test.

Suppose, $a_n, b_n \geq 0$ for $n \geq N$.

· $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ ($L = \infty$ allowed).

Then

a) if $L < \infty$ & $\sum_{n=1}^{\infty} b_n$ converges, then $\sum a_n$ converges.

b) if $L > 0$ & $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum a_n$ diverges.

Pf b) Under the conditions,

$\exists$ M such that $\forall n \geq M$, $\frac{a_n}{b_n} \geq \frac{L}{2} > c$

such that $a_n \geq c \, b_n$

$c > 0$ constant.

Now, for $n \geq N$ & M,

· $a_n, b_n \geq 0$

· $a_n \geq c \, b_n$

· $\sum_n c \, b_n = c \sum_n b_n$ diverges.

By comparism test, $\sum_{n=1}^{\infty} a_n$ diverges.

a) Similarly as b),

$\exists\, M \geq N$, such that $\forall n \geq M$ $\quad \frac{a_n}{b_n} \leq L+1$. , $L < \infty$.

Then $\displaystyle\sum_{n=M}^{\infty} a_n \leq \sum_{n=M}^{\infty} (L+1)\, b_n$

$a_n \geq 0$ for $n \geq M \geq N$.

$$= (L+1) \sum_{n=M}^{\infty} b_n < \infty$$

from $\displaystyle\sum_{n=M}^{\infty} b_n < +\infty$.

Thus $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. $\boxed{//}$

EX. $a_n = \dfrac{n^2 + 10^5 n + 2^{-n}}{n^4 + 2^{-n} - n^2}$.

$\displaystyle\sum_{n=1}^{\infty} a_n$ converge?

<sol>. See $\dfrac{n^2 + 10^5 n + 2^{-n}}{n^4 + 2^{-n} - n^2} \sim \dfrac{n^2}{n^4} = \dfrac{1}{n^2}$ for large $n \gg 1$.

So. compare it with $b_n = \frac{1}{n^2}$,

and check

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2 + 10^5 n + 2^{-n}}{n^4 + 2^{-n} - n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1 + 10^5 \frac{1}{n} + \frac{2^{-n}}{n}}{1 + \frac{2^{-n}}{n^4} - \frac{1}{n^2}} = 1$$

$< \infty$.

Thus, by limit comparison test,

(since $\displaystyle\sum_{n=1}^{\infty} b_n$ converges)

$\displaystyle\sum_{n=1}^{\infty} a_n$ converges. $\boxed{//}$

For comparison tests, we assume $a_n \geq 0$ for n large enough.
This does not restrict us much, because the following theorem.

Thm Suppose $\sum_{n=1}^{\infty} |a_n|$ converges.
Then $\sum_{n=1}^{\infty} a_n$ converges.

(proof). Consider $b_n = a_n + |a_n|$.

Then, $0 \leq b_n \leq 2|a_n|$

Thus $\sum_{n=1}^{\infty} b_n$ converges by comparing with $\sum_{n=1}^{\infty} 2|a_n|$.

But then
$$S_N = \sum_{n=1}^{N} a_n = \sum_{n=1}^{N} b_n - \sum_{n=1}^{N} |a_n|$$

both converge as $N \to \infty$.

Thus $\lim_{N \to \infty} S_N$ exists, so does $\sum_{n=1}^{\infty} a_n$ ▨

Absolute Convergence: $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, if $\sum_{n=1}^{\infty} |a_n|$ converges.

Conditional convergence: $\sum_{n=1}^{\infty} a_n$ is conditionally convergent
if $\sum_{n=1}^{\infty} a_n$ converges, but not $\sum_{n=1}^{\infty} |a_n|$.

In the Next lecture,
we will see examples of conditionally convergent series.

i.e. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$.

• There are examples that behave ROUGHLY like geometric series, but hard to apply comparison test directly.

e.g. ① $\sum_{n=1}^{\infty} \frac{n+1}{2^n}$, $\sum_{n=1}^{\infty} \frac{n!}{2^n}$, ...

② $\sum_{n=1}^{\infty} (1-\frac{1}{n})^{n^2}$, ...

Here, we may try to

① look at $\left|\frac{a_{n+1}}{a_n}\right|$ for large n. $\rightarrow$ Ratio test

② look at $\sqrt[n]{|a_n|}$ for large n $\rightarrow$ Root test.

Ratio test

For a given $\sum_{n=1}^{\infty} a_n$,

• IF $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ THEN $\sum_{n=1}^{\infty} a_n$ converges.

strict inequality

• IF $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ THEN $\sum_{n=1}^{\infty} a_n$ diverges.

(including the case $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$.) ← IMPORTANT!

Root test

For a given $\sum_{n=1}^{\infty} a_n$,

• IF $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, THEN $\sum_{n=1}^{\infty} a_n$ converges.

strict inequality

• IF $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ THEN $\sum_{n=1}^{\infty} a_n$ diverges.

(including the case $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.) ← IMPORTANT!

The idea for Ratio/Root tests

Let $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ (if the limit exists).

(or $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ (if the limit exists.)

Then, FOR VERY LARGE n (For convergence/divergence what matters is the behavior for larger n.)

(i.e. $n \geq K$ for a very large K)

$|a_{n+1}| \approx \rho |a_n|$, So, $\sum_{n=K}^{\infty} |a_n| \approx a_K \sum_{j=0}^{\infty} \rho^j$

i.e. the tail of $\sum_{n=1}^{\infty} |a_n|$

looks like a geometric series $\sum_{j=0}^{\infty} \rho^j$

note $\rho \geq 0$

since

$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, or $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

≥ 0.

- $\sum_{j=0}^{\infty} \rho^j$ converges if $0 \leq \rho < 1$
  diverges if $\rho > 1$.
- So, $\sum_{n=1}^{\infty} |a_n|$ converges if $\rho < 1$
  diverges if $\rho > 1$

the same for $\sum_{n=0}^{\infty} a_n$.

Check the rigorous proof in § 9.3.

Ex $\sum_{n=1}^{\infty} \frac{n!}{2^n}$.

: $a_n = \frac{n!}{2^n}$.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{2^{n+1}} \Big/ \frac{n!}{2^n} = \frac{(n+1)!}{n!} \Big/ \frac{2^{n+1}}{2^n} = \frac{(n+1)}{2}$$

$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1$.

So, by Ratio test, $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ diverges. $\boxed{}$.

Ex $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

(sol) $a_n = \left(1-\frac{1}{n}\right)^{n^2}$, $\sqrt[n]{a_n} = \left(\left(1-\frac{1}{n}\right)^{n^2}\right)^{\frac{1}{n}} = \left(1-\frac{1}{n}\right)^{n}$

$$\ln(1-x)^{\frac{1}{x}} = \frac{1}{x}\ln(1-x)$$

$$\lim_{x\to 0} \ln(1-x)^{\frac{1}{x}} = \lim_{x\to 0} \frac{-\frac{1}{1-x}}{1} \leftarrow \text{L'Hopital}$$

$$= -1.$$

$\therefore \lim_{x\to 0} (1-x)^{\frac{1}{x}} = e^{\lim_{x\to 0} \ln(1-x)^{\frac{1}{x}}} = e^{-1} = \frac{1}{e}$.

$\therefore \sqrt[n]{a_n} \xrightarrow[n\to\infty]{} \frac{1}{e} < 1$.

Thus, the series converges. $\boxed{}$

Note For $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n}$ the root test does not work, but we easily see it diverges since $\left(1-\frac{1}{n}\right)^n \to \frac{1}{e} \neq 0$.

※ In the case $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$.

we may NOT conclude. $\sum_{n=1}^{\infty} a_n$ diverges.

Since either cases may happen.

e.g.

- $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ but $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \xrightarrow[\text{as } n\to\infty]{} 1$ $\left(\sqrt[n]{a_n} \to 1 \text{ as } n\to\infty\right)$
- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges but $\frac{a_{n+1}}{a_n} = \frac{n^2}{n^2+1} \xrightarrow[\text{as } n\to\infty]{} 1$ $\left(\sqrt[n]{a_n} \to 1 \text{ as } n\to\infty\right)$

Checked these in Lec 32,

by comparing to integral $\int_1^{\infty} \frac{1}{x^p}\,dx$, $p=1,2$.

Ratio test is NOT useful if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.
Root test " NOT " if $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$.

e.g. For $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$, $a_n = \frac{n^2+1}{n^3+1}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2+1}{(n+1)^3+1} \Big/ \frac{n^2+1}{n^3+1}$$

$$= \frac{(n+1)^2+1}{n^2+1} \Big/ \frac{(n+1)^3+1}{n^3+1} \xrightarrow[\text{as } n\to\infty]{} 1/1 = 1$$

MORE EXAMPLES.

Ex Define a function $f(x)$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{1+n}{n^3 2^n} x^n .$$

typo corrected

Find all x where $f(x)$ is well-defined.

i.e. those x where the series converges.

<sol>. • $a_n = \dfrac{1+n}{n^3 2^n} x^n$

involves exponent changing in n.

natural to try ratio test.

• $$\frac{a_{n+1}}{a_n} = \left[\frac{1+n+1}{(n+1)^3 2^{n+1}} \cdot x^{n+1}\right] \Big/ \left[\frac{1+n}{n^3 2^n} x^n\right]$$

$$= \frac{1+n+1}{1+n} \cdot \frac{n^3}{(n+1)^3} \cdot \frac{x}{2}$$

• $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1+n+1}{1+n} \cdot \frac{n^3}{(n+1)^3} \cdot \frac{|x|}{2}.$$

• Consider $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \frac{|x|}{2} \cdot \lim_{n\to\infty} \left\{\left[\frac{1+n+1}{1+n}\right]\left[\frac{n^3}{(n+1)^3}\right]\right\}$$

$$= \frac{|x|}{2} \cdot \underbrace{\left(\lim_{n\to\infty} \frac{1+n+1}{1+n}\right)}_{=1} \cdot \underbrace{\left(\lim_{n\to\infty} \frac{n^3}{(n+1)^3}\right)}_{=1}$$

$$= \frac{|x|}{2}$$

$\bullet \lim_{n\to\infty} \frac{1+n+1}{1+n}$

$= \lim_{n\to\infty} \frac{\cancel{n}(\frac{1}{n}+1+\frac{1}{n})}{\cancel{n}(1+\frac{1}{n})}$

$= 1$

$\bullet \lim_{n\to\infty} \frac{n^3}{(n+1)^3}$

$= \lim_{n\to\infty} \frac{\cancel{n^3}}{\cancel{n^3}(1+\frac{1}{n})^3}$

$= 1$

- So, by Ratio test, $\sum_{n=1}^{\infty} \frac{1+n}{n^3 2^n} x^n$
  - converges for $\frac{|x|}{2}<1$. i.e. for $|x|<2$
  - diverges for $\frac{|x|}{2}>1$ i.e. for $|x|>2$.

- For $x=2$, $x=-2$?

$\cdot x=2$: $\sum_{n=1}^{\infty} \frac{1+n}{n^3 2^n} 2^n = \sum_{n=1}^{\infty} \frac{1+n}{n^3}$

It behaves like $\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$. guess it converges.

In fact, $\cdot$ $\frac{1+n}{n^3} < \frac{2n}{n^3} = \frac{2}{n^2}$ for $n \geq 1$

$\cdot$ $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges by integral test.

(We knew this from Lec 32)

So, by Comparison test,

$\sum_{n=1}^{\infty} \frac{1+n}{n^3}$ converges. So $x=2$ is chosen.

$\cdot x=-2$: $\sum_{n=1}^{\infty} \frac{1+n}{n^3 2^n} (-2)^n = \sum_{n=1}^{\infty} \frac{1+n}{n^3} (-1)^n$.

Note. $\left| \frac{1+n}{n^3} (-1)^n \right| = \frac{1+n}{n^3} < \frac{2n}{n^3}$ for $n \geq 1$.

$\cdot$ $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges.

So, by the comparison test $\sum_{n=1}^{\infty} \frac{1+n}{n^3 2^n} (-2)^n$ converges

so. choose $x=-2$.

- The series converges for $-2 \leq x \leq 2$ only. $\boxed{}$

including $x=2,-2$.

Convergence / divergence for $\sum_{n=0}^{\infty} a_n$

"strategy".

Try

⓪ see "rough" behavior, make guesses.
(e.g. Identify dominating terms).

① divergence test : · see whether $a_n \to 0$
- If not, $\sum_{n=0}^{\infty} a_n$ diverges

② If a_n looks complicated
: Find a simpler, but related series $\sum_{n=0}^{\infty} b_n$

try comparison test (Including · limit comparison · ratio test · root test)

(usually, want · $|a_n| \leq b_n$ for all $n \geq M$
or · $a_n \geq b_n$ for all $n \geq M$

③ For not so complicated a_n
(e.g. $\sum_{n=0}^{\infty} b_n$ found in ②)

try - integral test : for some monotonically decreasing a_n.
involving n^p, $\ln n$, etc.

- ratio test : for a_n involving r^n, $n!$, etc.

Ex. Check convergence : $\sum_{n=1}^{\infty} \frac{n^2+\sin(n)}{n!+e^{-n}}$

<sol>.

- By considering dominating terms n^2, $n!$

  We see :
  The sequence $a_n = \frac{n^2+\sin(n)}{n!+e^{-n}}$ behaves like $\frac{n^2}{n!}$

- $n!$ grows much faster than n^2.

$$n! = 1\cdot 2\cdot 3 \cdots \cdots (n-3)(n-2)(n-1)\,n$$

many factors growing like n

$$n^2 = n \cdot n$$

only two factors growing like n.

So, can guess convergence of $\sum_{n=1}^{\infty} \frac{n^2}{n!}$,

so, also for $\sum_{n=1}^{\infty} \frac{n^2+\sin(n)}{n!+e^{-n}}$.

- Find a simpler but related sequence :

$$0 \le \frac{n^2+\sin(n)}{n!+e^{-n}} \le \frac{n^2+1}{n!} \le \boxed{\frac{n^2}{n!} = b_n}$$

for $n \ge 2$

$\sin(n) \le 1$
$e^{-n} > 0$

name

At this moment,
  We want to use Comparison test.

- For convergence of $\sum_{n=1}^{\infty} \frac{n^2}{n!} = \sum_{n=1}^{\infty} b_n$ ← name.

apply ratio test. (natural because we see $n!$).

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^2}{(n+1)!} \Big/ \frac{n^2}{n!}$$

$$= \frac{(n+1)^2}{n^2} \Big/ \frac{(n+1)!}{n!}$$

$$= \frac{(n+1)^2}{n^2} \Big/ n+1$$

$$= \frac{n+1}{n^2} \longrightarrow 0 < 1 \quad \text{as } n \to \infty.$$

So, by ratio test,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n^2}{n!} \quad \text{converges.}$$

- Now for the original series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2 + \sin(n)}{n! + e^{-n}}$

$$\left\{ \begin{array}{l} \bullet\ |a_n| \leq b_n \quad n \geq 2 \\ \bullet\ \sum_{n=1}^{\infty} b_n \text{ converges} \end{array} \right. \xRightarrow{\boxed{\text{Comparison test}}} \sum_{n=1}^{\infty} a_n \ \underline{\underline{\text{converges}}}. \ \blacksquare$$