Lec 36. Series.

Today: §9.2.
- series
- convergence / divergence of series.
  - Basic examples

Fri: §9.2 ~ 9.3

- Methods to check convergence/divergence
  - comparision test
  - divergence test

Next Mon: (§ 9.3~9.4)

  - Comparison to integrals (Integral test)
  - ratio test. / root test.
  - estimates on the sums.

Today:

- Definition of series: $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots\cdots$
- convergence / divergence
  - Does the sum of infinitely many numbers make sense?
- A few basic examples
  - geometric series: $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$
  - telescoping series: e.g. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots$
  - Harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$
- If time permits: comparison test.

§ Series : examples

(infinite) series : sum of infinitely many terms

$$a_0 + a_1 + a_2 + a_3 + \cdots$$

$$= \sum_{n=0}^{\infty} a_n \quad \leftarrow \text{notation}$$

(a series can start with other n, e.g. $n=2$)

e.g. $1+2+3+4+5+6+\cdots = \sum_{n=0}^{\infty} n$

e.g. $1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}$

e.g. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots\cdots = \sum_{n=1}^{\infty} \frac{1}{n}$

e.g. $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$

e.g. $1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$ (In fact the sum is e)

Here,

$$\begin{cases} k! = 1\cdot 2\cdot 3\cdots(k-1)k. & \text{e.g. } 2! = 1\cdot 2 = 2. \quad 3! = 1\cdot 2\cdot 3 = 6. \\ 0! = 1 \\ 1! = 1 \end{cases}$$

• Background: Where they appear / why important?

- many (scientific) computations require "series"

e.g. Computers use series expression of functions.

For example,

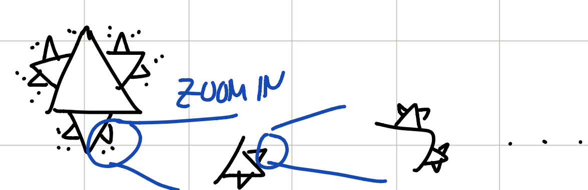
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

(We will learn more details in §9.6. Taylor series.)

• There are also many cases one wants to add infinitely many terms.

e.g. area of "fractal shapes"



The area is the "infinite sum" of areas of the triangles

$$T_1 + T_2 + T_3 + \cdots$$

- The most basic question:

· Does the sum of infinitely many terms make sense?

Def (Convergence)

A series $\sum_{k=0}^{\infty} a_k$ is said to be $\boxed{\text{Convergent}}$

if the partial sum $S_N = \sum_{k=0}^{N} a_k$ converges as $N \to \infty$.

- partial sum $S_N = \sum_{k=0}^{N} a_k$ gives a sequence

$S_0 = a_0$

$S_1 = a_0 + a_1$

$S_2 = a_0 + a_1 + a_2$

$\vdots$

- The value of $\sum_{k=0}^{\infty} a_k$

$$\sum_{k=0}^{\infty} a_k = \lim_{N \to \infty} \sum_{k=0}^{N} a_k$$

- $\sum_{k=0}^{\infty} a_k$ is convergent by definition

if $\lim_{N \to \infty} S_N = \lim_{N \to \infty} \sum_{k=0}^{N} a_k$ exists.

Def (Divergence)

$\sum_{k=0}^{\infty} a_k$ diverges (or does NOT converge)

by definition, if the sequence $S_N = \sum_{k=0}^{N} a_k$ diverges as $N \to \infty$

Rmk For convergence/divergence of a series,
the first finite number of terms ("Head") do not matter.

e.g. $$\underbrace{10^{16} + 10^{10^2} + 10^{10^3} + \cdots + 10^{10^{100}}}_{\text{For convergence, the Head (How large / How many) does not matter.}}^{\text{the first } 10^{100} \text{ terms}} + \underbrace{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots}_{\text{For convergence question the behavior of the tail matters.}}$$

$\boxed{\sum_{k=0}^{\infty} a_k \text{ converges / diverges if } \sum_{k=M}^{\infty} a_k \text{ converges / diverges for an } M}$

Given a series $a_0 + a_1 + a_2 + a_3 + \cdots = \sum_{k=0}^{\infty} a_k$

the first thing to consider is

whether it converges or diverges.

- After that, one may ask
  - what value it has
  - how to estimate its value.
  - etc ....

Rmk When we consider functions represented by a series like,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

an important question is where the series expression converges. (for which x).

- Basic examples.

For some (simple) series, the value of the sum is computable.

- e.g. (geometric series)

For $a \neq 0$, $r \neq 0$.

$$\sum_{k=0}^{\infty} a r^k = a + ar + ar^2 + ar^3 + \cdots$$

$\leftarrow \frac{a_{k+1}}{a_k} = r$.

Partial sum $S_N = \sum_{k=0}^{N} a r^k = a \sum_{k=0}^{N} r^k$

$$= \begin{cases} a(N+1) & \text{if } r = 1. \\ a\left(\frac{1 - r^{N+1}}{1-r}\right) & \text{if } r \neq 1 \end{cases}$$

from earlier lectures.

$$\therefore \sum_{k=0}^{\infty} a r^k = \lim_{N\to\infty} S_N = \lim_{N\to\infty} a \sum_{k=0}^{\infty} r^k$$

$$= \begin{cases} a\cdot\left(\lim_{N\to\infty}(N+1)\right) & \text{if } r=1 \\ a\cdot\left(\lim_{N\to\infty}\frac{1-r^{N+1}}{1-r}\right) & \text{if } r\neq 1 \end{cases}$$

So, $\sum_{k=0}^{\infty} a r^k$ $\begin{cases} \text{diverges to } +\infty \text{ if } r \geq 1 \text{ \& } a>0 \\ \qquad\qquad\;\; -\infty \text{ if } r\geq 1 \text{ \& } a<0 \\ \text{diverges if } r\leq -1 \\ \text{converges \& its value is } \frac{a}{1-r} \text{ if } |r|<1 \end{cases}$

e.g. (telescoping series)

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{k(k+1)}$$

$$= \lim_{N\to\infty}\sum_{k=1}^{N}\left[\frac{1}{k}-\frac{1}{k+1}\right]$$

easy partial fraction.

$$\frac{1}{k(k+1)} = \frac{1}{k}-\frac{1}{k+1}$$

$$= \lim_{N\to\infty}\left[\left(1-\cancel{\tfrac{1}{2}}\right)+\left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right)+\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{4}}\right)+\cdots+\left(\cancel{\tfrac{1}{N-1}}-\cancel{\tfrac{1}{N}}\right)+\left(\cancel{\tfrac{1}{N}}-\tfrac{1}{N+1}\right)\right]$$

$$= \lim_{N\to\infty}\left[1-\frac{1}{N+1}\right]$$

cancellations between consecutive terms

$= 1.$ The series converges & its value is 1. ▨

Other telescoping series

Ex $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+3)}$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{(k+1)(k+3)}$$

partial fraction

$$\frac{1}{(k+1)(k+3)} = \frac{1}{2}\left[\frac{1}{k+1} - \frac{1}{k+3}\right]$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{2}\left[\frac{1}{k+1} - \frac{1}{k+3}\right]$$

$$= \lim_{N\to\infty} \frac{1}{2} \sum_{k=1}^{N} \left[\frac{1}{k+1} - \frac{1}{k+3}\right]$$

cancellation between nearby terms.

remains

$$= \lim_{N\to\infty} \frac{1}{2}\Big[\frac{1}{2} - \cancel{\frac{1}{4}} + \frac{1}{3} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} + \cancel{} \cdots$$

$$\cancel{\frac{1}{N-1}} - \cancel{\frac{1}{N+1}} + \cancel{\frac{1}{N}} \underbrace{- \frac{1}{N+2}}_{\text{remains}} + \cancel{\frac{1}{N+1}} \underbrace{- \frac{1}{N+3}}_{\text{remains}}\Big]$$

$$= \lim_{N\to\infty} \frac{1}{2}\left[\frac{1}{2} + \frac{1}{3} - \frac{1}{N+2} - \frac{1}{N+3}\right]$$

$$\lim_{N\to\infty} \frac{1}{N+1} = 0$$

$$\lim_{N\to\infty} \frac{1}{N+3} = 0.$$

$$= \frac{1}{2}\cdot\left(\frac{1}{2} + \frac{1}{3}\right)$$

$$= \frac{1}{2}\cdot\frac{3+2}{6} = \frac{5}{12}$$

The series converges & its value is $\frac{5}{12}$. ▨

- Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

· An important divergent series.

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k} = \infty} \leftarrow \text{i.e. the series diverges to } \infty.$$

Reason: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

regroup

$$= \left(1+\frac{1}{2}\right) + \left(\frac{1}{3}+\frac{1}{2^2}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{2^3}\right) + \left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{2^4}\right)$$

$$+ \left(\frac{1}{17}+\cdots+\frac{1}{2^5}\right) + \left(\frac{1}{2^5+1}+\cdots+\frac{1}{2^6}\right) + \cdots\cdots$$

The partial sum up to 2^n-th term:

$$S_{2^n} = 1 + \underbrace{\frac{1}{2}} + \Big(\underbrace{\overbrace{\frac{1}{3}+\frac{1}{2^2}}^{2\text{-terms}}}\Big) + \Big(\underbrace{\overbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{2^3}}^{2^2 \text{ terms}}}\Big) + \cdots + \Big(\overbrace{\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}}^{2^{n-1} \text{ terms}}\Big)$$

There are n such groups & in each group:

· $\frac{1}{3} + \frac{1}{2^2} > \frac{1}{2^2} + \frac{1}{2^2} = 2 \cdot \frac{1}{2^2} = \frac{1}{2}$

· $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{2^3} > \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3} = 2^2 \cdot \frac{1}{2^3} = \frac{1}{2}$

$\vdots$

· $\underbrace{\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^n}}_{2^{n-1} \text{ terms}} > \underbrace{\frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}}_{2^{n-1} \text{ terms}} = 2^{n-1} \cdot \frac{1}{2^n} = \frac{1}{2}$

So, $S_{2^n} > 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{n \text{ terms}} = 1 + \frac{n+1}{2}$.

$\therefore \quad S_{2^n} > 1 + \frac{n}{2}$

So $\lim_{n\to\infty} S_{2^n} = \infty$ since $\lim_{n\to\infty}\left(1+\frac{n}{2}\right) = \infty$.

Thus $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$. ▨

For convergence of a series $\sum_{n=0}^{\infty} a_n$,
One can FIRST try

- Divergence test: look at what happens to a_n as $n\to\infty$

Divergence test

$$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=0}^{\infty} a_n \text{ diverges.}$$

WARNING

$$\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=0}^{\infty} a_n \text{ converges}$$

WRONG.

e.g. $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (diverges), but $\lim_{n\to\infty} \frac{1}{n} = 0$

• <u>Reason</u> For the partial sum $S_N = \sum_{n=0}^{N} a_n$

$a_N = S_N - S_{N-1}$.

So if $\sum_{n=0}^{\infty} a_n$ converges, say, to L

then $\lim_{N\to\infty} S_N = L = \lim_{N\to\infty} S_{N-1}$

So $\lim_{N\to\infty} a_N = \lim_{N\to\infty} S_N - \lim_{N\to\infty} S_{N-1} = L - L = 0$.

So, the condition $\lim_{N\to\infty} a_N = 0$

is <u>NECESSARY</u>

for convergence of $\sum_{n=0}^{\infty} a_n$.

In other words, if $\lim_{N\to\infty} a_N = 0$ is <u>violated</u>

then $\sum_{n=0}^{\infty} a_n$ does <u>NOT</u> converge

i.e. it diverges.

e.g. $\sum_{n=1}^{\infty} (-1)^n$ diverges

since $\lim_{n \to \infty} (-1)^n$ does NOT exist.

e.g. $\sum_{n=1}^{\infty} \frac{n}{n+1}$.

$\lim_{n \to \infty} \frac{n}{n+1} = 1 \neq 0$ so, $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

e.g. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges because

$a_n = \frac{n^n}{n!} = \frac{n}{n} \cdot \frac{n}{n-1} \cdot \frac{n}{n-2} \cdot \cdots \frac{n}{2} \cdot \frac{n}{1} \to \infty$ as $n \to \infty$.

for $\frac{n^n}{n!} \geq 1 \cdot 1 \cdot \cdots \cdot 1 \cdot n \geq n \to \infty$ as $n \to \infty$.

After knowing some basic examples

one can consider convergence/divergence

of more complicated series

by comparing those to the known & simpler series.

- "Comparison test"

Thm (Comparison test) (divergent case) Given $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$,

- $\sum_{n=0}^{\infty} a_n = \infty$ if $\begin{cases} \cdot\ a_n \geq b_n \text{ for all } n \geq M \text{ for some } M. \\ \sum_{n=0}^{\infty} b_n = \infty \end{cases}$

- $\sum_{n=0}^{\infty} a_n = -\infty$ if $\begin{cases} a_n \leq b_n \text{ for all } n \geq M \text{ for some } M \\ \sum_{n=0}^{\infty} b_n = -\infty. \end{cases}$

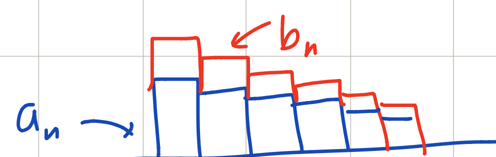
Thm (Comparison test) (convergent case) Given $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$,

$\sum_{n=0}^{\infty} a_n$ converges if
$$\begin{cases} |a_n| \le b_n \text{ for all } n \ge M \text{ for some } M. \\ \sum_{n=0}^{\infty} b_n \text{ converges} \end{cases}$$

reason

Case $0 \le a_n \le b_n$ for all $n \ge 0$ & $\sum_{n=0}^{\infty} b_n$ converges.

If $b_n \ge a_n$. for $n \ge 0$



then $\sum_{n=0}^{N} a_n \le \sum_{n=0}^{N} b_n$ for large N.

If $\sum_{n=0}^{\infty} b_n$ converges & $\sum_{n=0}^{\infty} b_n = L$

then for $S_N = \sum_{n=0}^{N} a_n$,

$$S_N \le S_{N+1} \le S_{N+2} \le \cdots \le \sum_{n=0}^{\infty} b_n = L$$

adding nonnegative terms a_n. the sum increases.

S_N is a monotone sequence and bounded.

So, $\lim_{N \to \infty} S_N$ exists.

So $\sum_{n=0}^{\infty} a_n = \lim_{N \to \infty} S_N$ converges. $\square$

- How to use comparison test.

Idea: for comparison

- Identify the dominant term (most rapidly increasing)
- Try to find a simpler series with the same dominant term. where the convergence/divergence is easier.
- Compare the original series with the simpler series.

e.g. Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+e^{-n}}$ converge?

Solution: Next lecture.