Lec35_Part2 Sequences

Iterated maps. . discrete time dynamical systems

- stability of fixed points

$\boxed{\text{The dynamical behavior of logistic map, } x_{t+1} = g(x_t), \; g(x) = \alpha x(1-x), \; x \geq 0. \text{ heavily depends on the value of } \alpha \geq 0.}$

• Fixed points

Solve $x = g(x)$ ← fixed points are where $y = g(x)$ intersects with $y = x$.

$\Rightarrow x = 0$ or $1 = \alpha(1-x)$

$\Leftrightarrow 1 = \alpha - \alpha x$

$\alpha x = \alpha - 1$

Fixed points (equilibrium) $\boxed{\therefore \; x = 0 \text{ or } x = \frac{\alpha - 1}{\alpha} = 1 - \frac{1}{\alpha}}$

$\boxed{\text{Carrying capacity } x^* = 1 - \frac{1}{\alpha}}$

for the population model given by the logistic map.

Note in a population model $x \geq 0$.

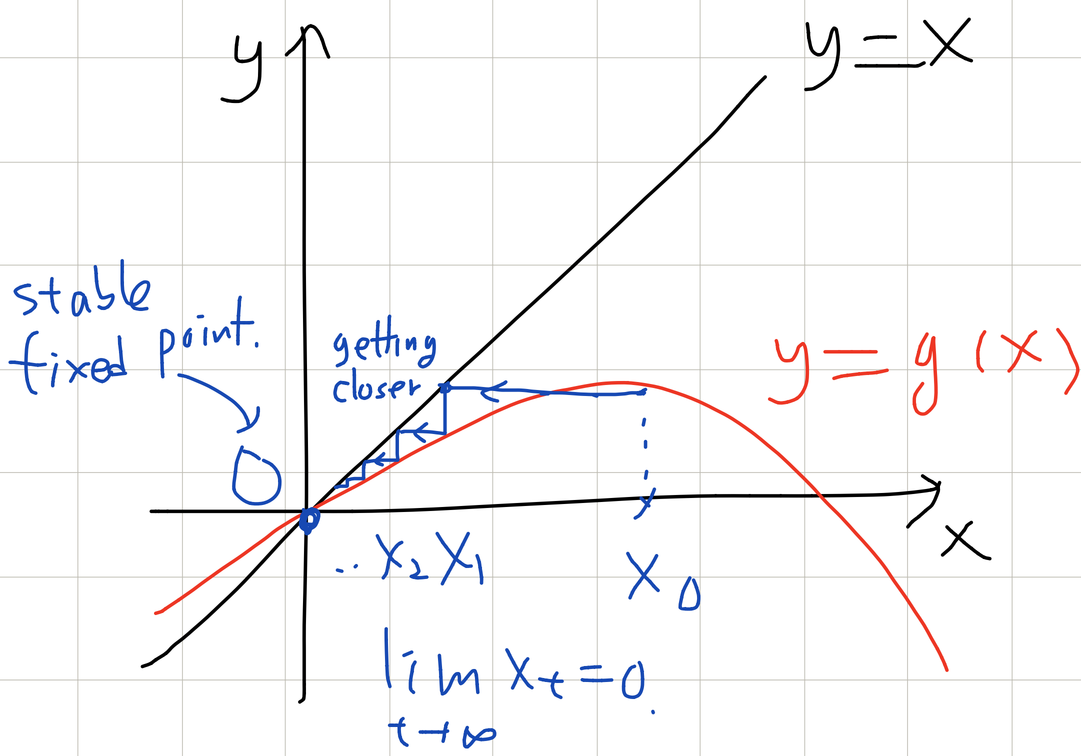
$x^* < 0$ is not meaningful.

But, if $0<\alpha<1$, then $x^* = 1-\frac{1}{\alpha} < 0$.

So, if $0<\alpha<1$, then only one meaningful fixed point $x=0$ in $x \geq 0$.

$\boxed{\text{Case } 0<\alpha<1.}$ only one Fixed point. in $x \geq 0$.

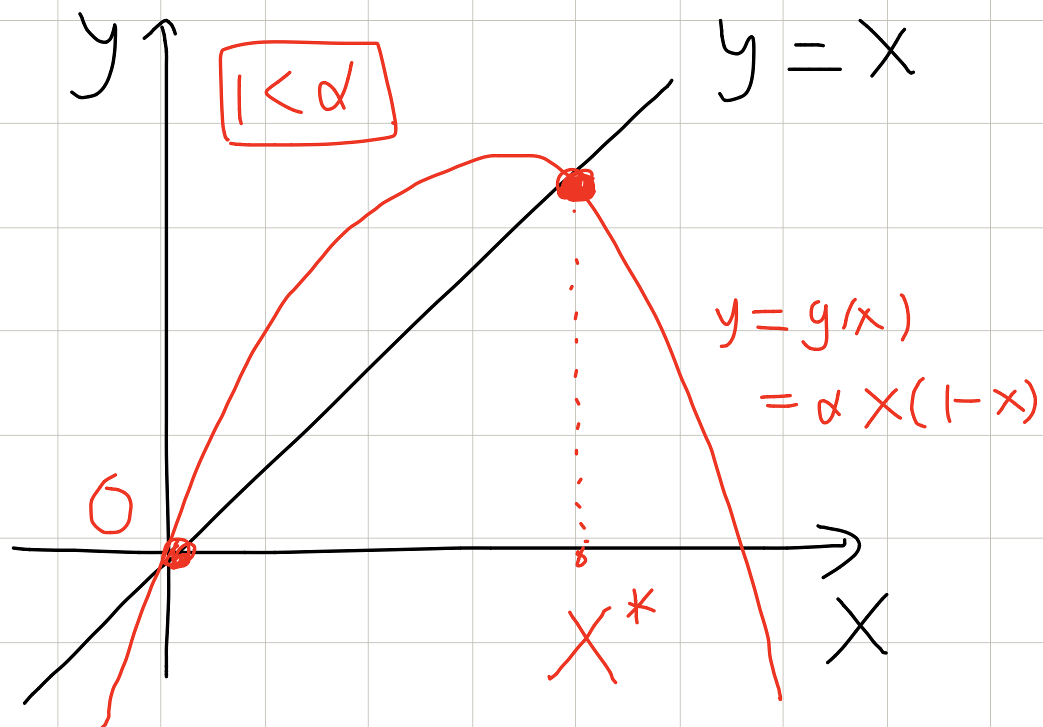
$\lim_{t\to\infty} x_t = 0$ The specis will eventually be extinct



- $\boxed{\text{Case } 1<\alpha}$.

two fixed points

$x=0$, $x=x^*=1-\frac{1}{\alpha}$.



- Stability of fixed points.

- A test to check stability

$\bar{v}$ a fixed point, $g(\bar{v}) = \bar{v}$

strict inequalities

- **stable** if $|g'(\bar{v})| < 1$.
- **unstable** if $|g'(\bar{v})| > 1$

- Stability at fixed points : check $g'(x)$.

$$g(x) = \alpha x(1-x) = \alpha x - \alpha x^2$$

$$g'(x) = \alpha - 2\alpha x .$$

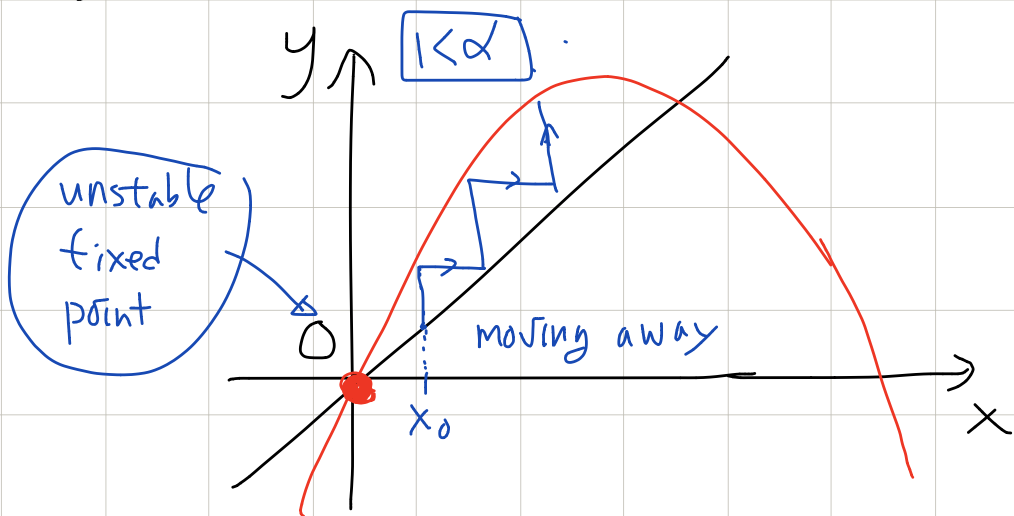
Check stability in the case $\alpha > 1$

Two fixed points $x = 0$, $x = x^* = 1 - \frac{1}{\alpha}$.

At $x=0$: $g'(0) = \alpha$ & In our case $\alpha > 1$.

$\therefore g'(0) > 1$.

So $x=0$ is unstable.

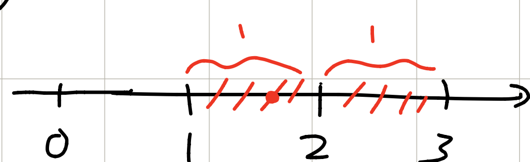


- **At $x = 1-\frac{1}{\alpha}$**:

$$g'(1-\tfrac{1}{\alpha}) = \alpha - 2\alpha(1-\tfrac{1}{\alpha})$$
$$= \alpha - 2\alpha + 2$$

$$\therefore g'(\alpha) = 2-\alpha$$

$$|g'(\alpha)|<1 \iff -1 < 2-\alpha < 1 \iff 1<\alpha<3.$$



$\therefore$ Stable if $1<\alpha<3$

unstable if $\alpha > 3$.

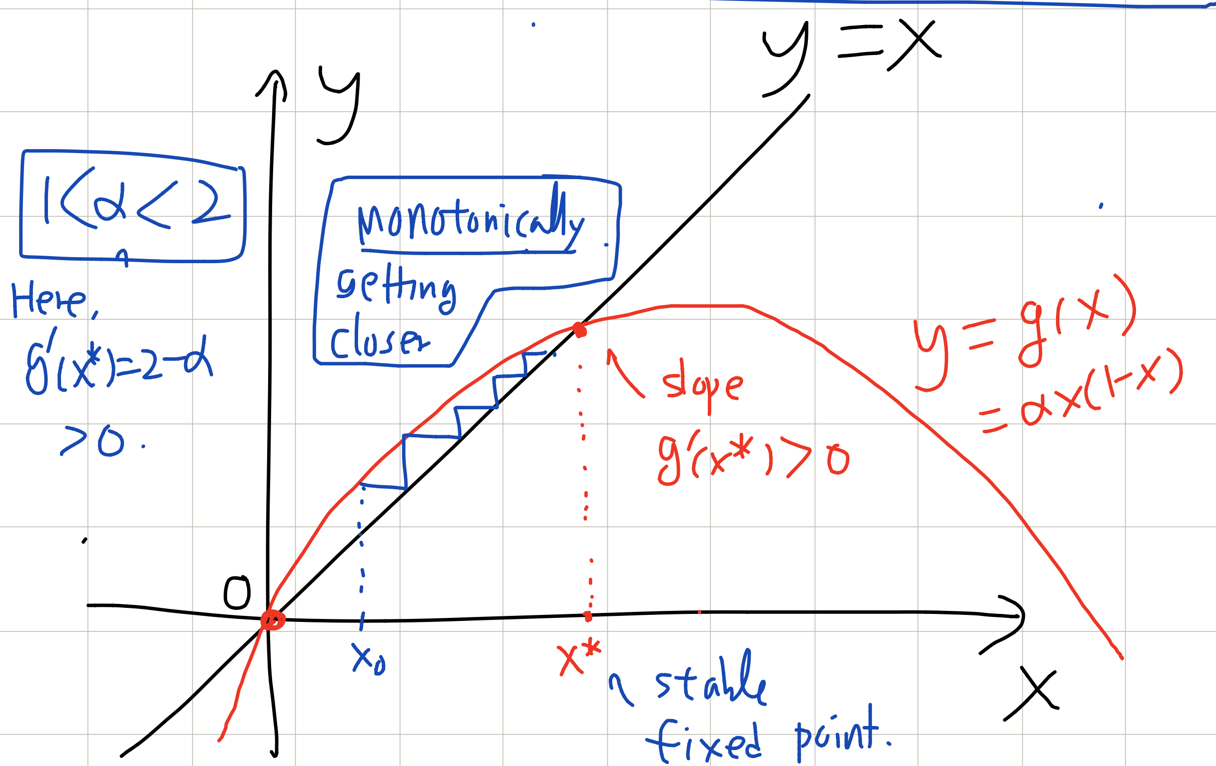
- $X = X^* = 1 - \frac{1}{\alpha}$

is a stable fixed point for $1 < \alpha < 3$

Sign of $g'(x^*)$ further determines how sequences converge to x^*.

- If x^* is stable fixed point of $g(x)$, and $g'(x^*) > 0$

then NEAR x^*, sequences converges to x^* MONOTONICally.



Thus, for $1 < \alpha < 2$ (this is the condition Δs is small in the previous discussion.)

the logistic map

shows a similar behavior

as the logistic differential eqn.

"Population converges monotonically to

the carrying capacity".

HOWEVER, as we see below

If $2 < \alpha$, then

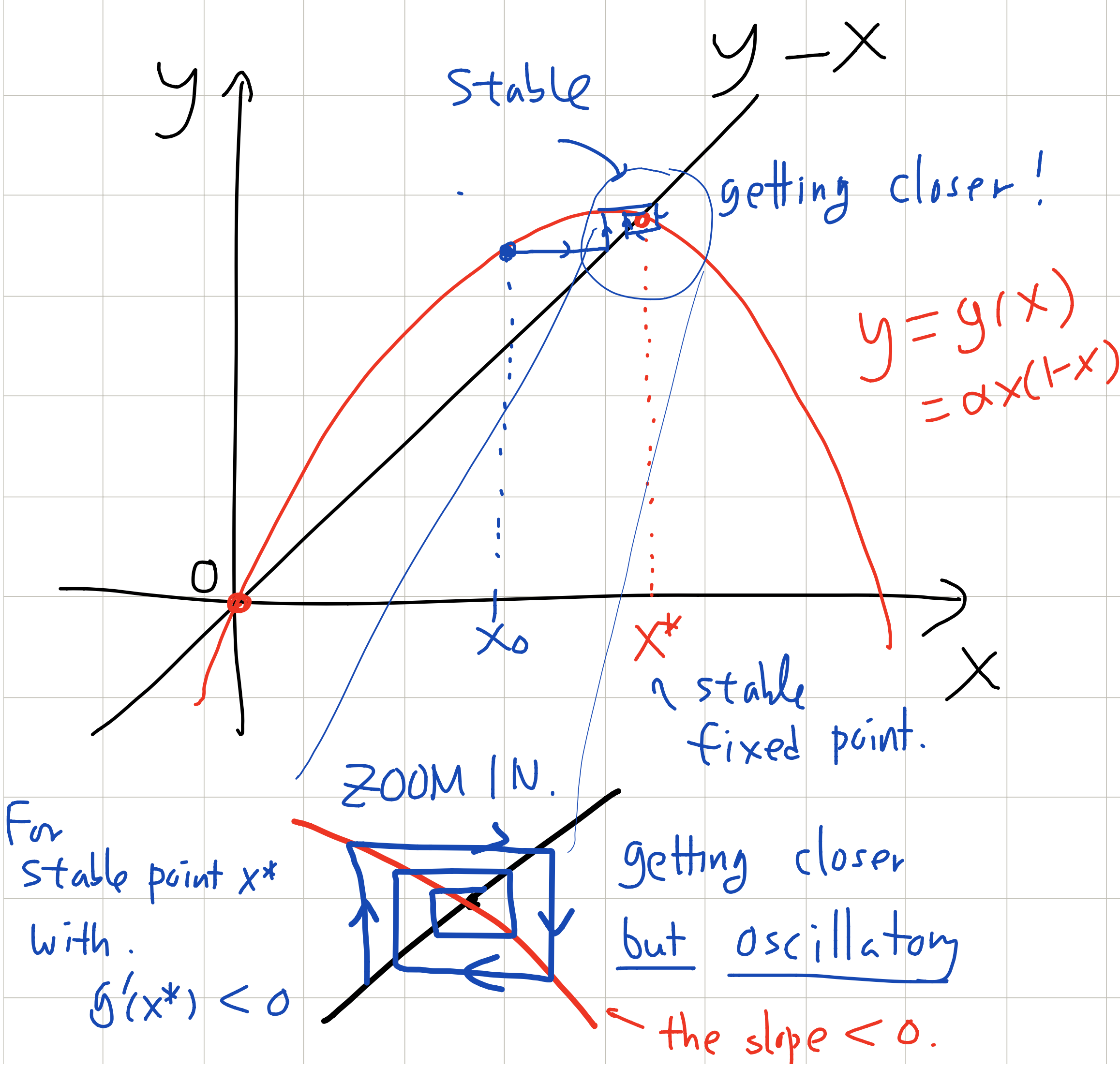
the logistic map demonstrate

a different (& more interesting)

behavior from the logistic differential eqn.

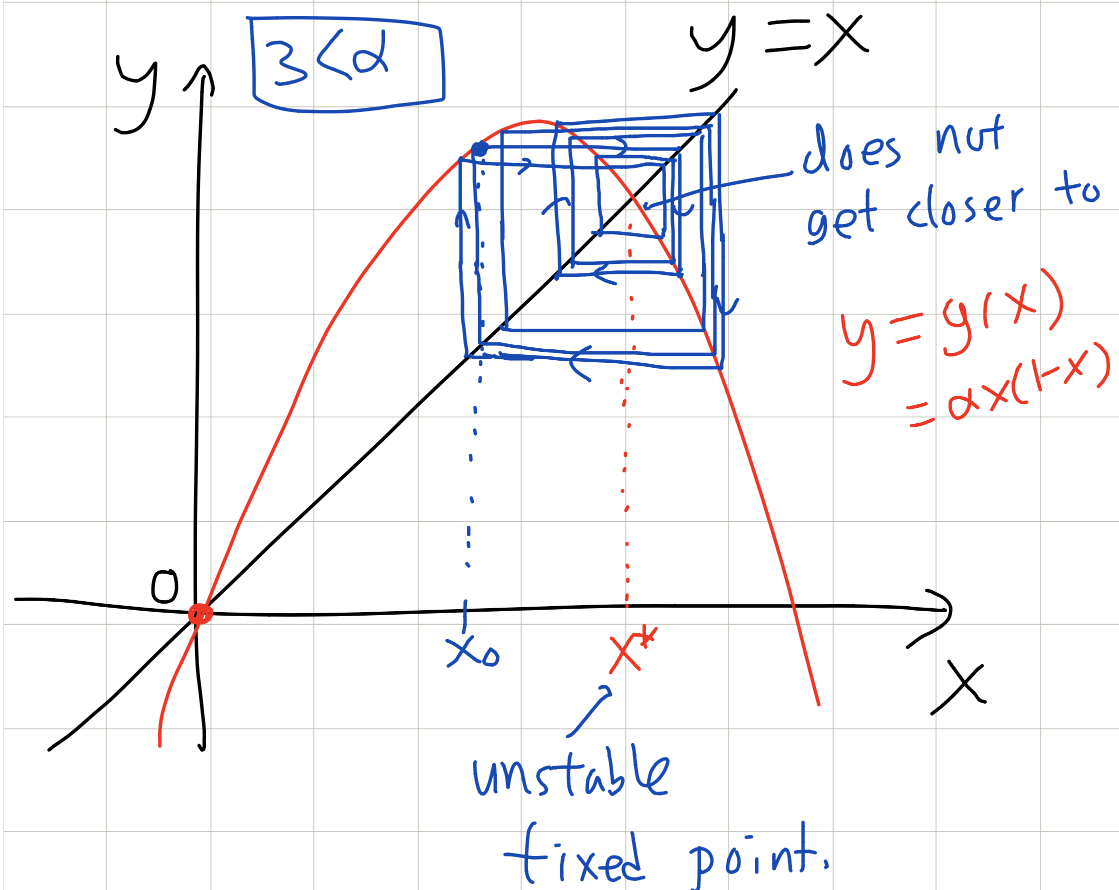
- If x^* is stable fixed point of $g(x)$.
  and $g'(x^*) < 0$

  then NEAR x^*, convergence to x^* is oscillatory.



UNSTABLE CASE.

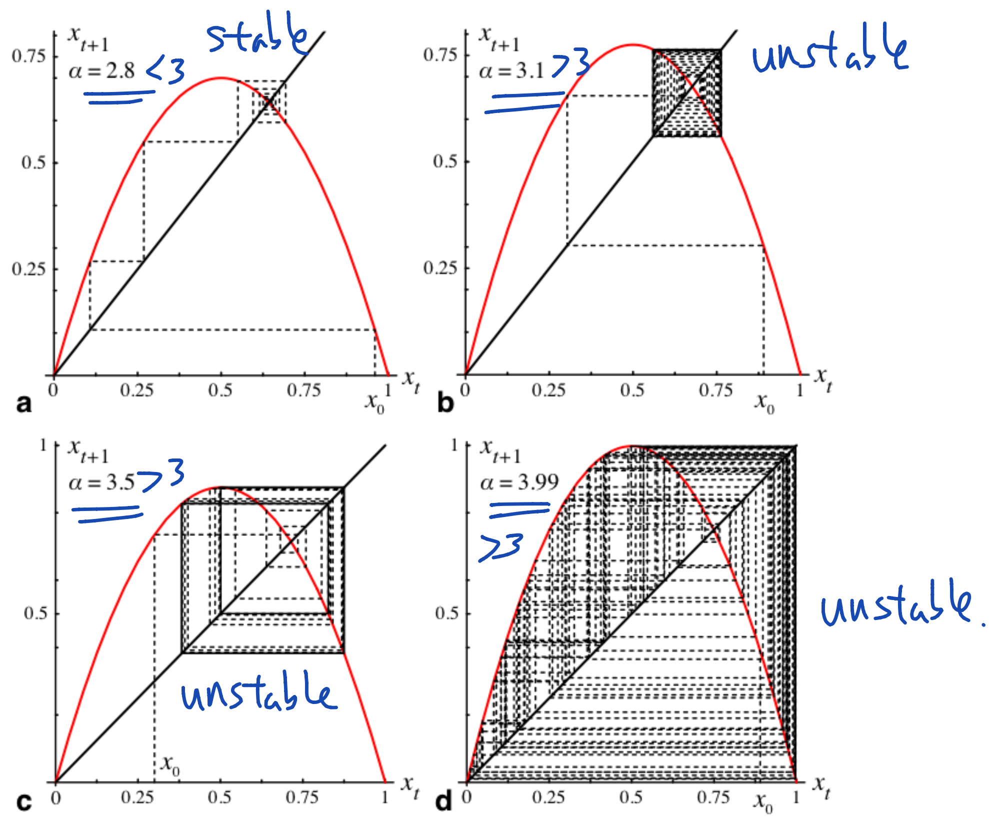
- $X^* = 1 - \frac{1}{\alpha}$ is <u>NOT</u> stable for $\alpha > 3$.



As in this picture,

unstable fixed points

MAY have periodic orbits around it.



For $\alpha > 3$ gets larger

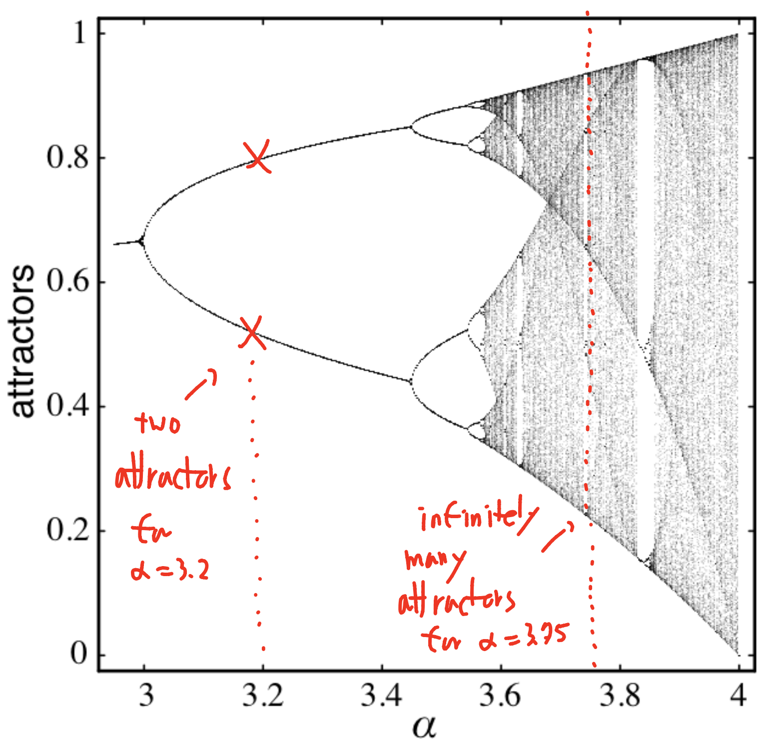
more complicated behavior (e.g. very long periodic orbits

• MANY periodic orbits

• unpredictable sequences, etc.

appears.

• Chaotic behavior to change of α.



Next lecture:

Series. §9.2.