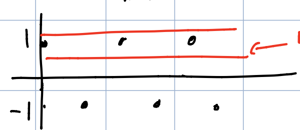
Lec 3X

- Sequences.
  - convergence.

§ 2.1.
§ Appendix III. (A 22 ~ 23)

EX. Show that $\{(-1)^n\}_{n=1}^{\infty}$ diverges (= does NOT converge).

(sol) Obvious:



no way to contain the sequence in a small ϵ window.

Formal Proof by contradiction:

Suppose $\lim_{n\to\infty} a_n = L$ for $a_n = (-1)^n$.

Choose $\epsilon = \frac{1}{10}$.

Then $\exists\ N$ s.t. $\forall\ n \geq N$ $|a_n - L| < \frac{1}{10}$.

But
$$|a_n - L| = |(-1)^n - L| = \begin{cases} |L+1| & n = \text{odd} \\ |L-1| & n = \text{even} \end{cases}$$

For $|L+1| < \frac{1}{10} \Rightarrow -\frac{1}{10} + 1 < L < 1 + \frac{1}{10}$ — ①

For $|L-1| < \frac{1}{10} \Rightarrow -\frac{1}{10} - 1 < L < -1 + \frac{1}{10}$ — ②

No such L satisfying ① & ② exists.

Contradiction to $\lim_{n\to\infty} a_n = L$.

$\therefore$ The limit does NOT exists. Diverges. $\blacksquare$

§ Theorems about convergence of sequences.

Thm If $\{a_n\}$ converges, then it is bounded. (i.e. $\exists$ $M > 0$ such that $|a_n| \leq M$ $\forall n$.)

(proof) Suppose $\lim_{n\to\infty} a_n = L$. Then, $\exists$ N s.t.
$$L - 1 \leq a_n \leq L + 1 \quad \forall\ n \geq N.$$

Now, let $M = \max\{|a_1|, \cdots, |a_{N-1}|, |a_N|, |L|+1\}$,
then $|a_n| \leq M$ $\forall n \geq 1$. $\blacksquare$

Thm $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B \Rightarrow \lim_{n\to\infty} \begin{cases} a_n b_n \\ a_n \pm b_n \\ \frac{a_n}{b_n} \end{cases}$ resp. $= \begin{cases} AB \\ A\pm B \\ \frac{A}{B} \end{cases}$.

assuming $b_n \neq 0$, $B \neq 0$

(proof). ※ This is exactly the same proof for limits of functions.

- $\lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} b_n = B$

$\Rightarrow \forall \varepsilon_1 > 0, \exists N_1$ s.t. $\forall n \geq N_1$

$|a_n - A| < \varepsilon_1$, $|b_n - B| < \varepsilon_1$.

Note: $a_n b_n = (a_n - A + A)(b_n - B + B)$

$= (a_n - A)\cdot(b_n - B) + (a_n - A)B + A(b_n - B) + AB$.

$\therefore a_n b_n - AB = (a_n - A)(b_n - B) + (a_n - A)B + A(b_n - B)$

- $\forall \varepsilon$, choose ε_1 & N_1 s.t. $|a_n - A| \leq \varepsilon_1$, $|b_n - B| \leq \varepsilon_1$ $\forall n \geq N_1$

with $\varepsilon_1^2 + \varepsilon_1(A+B) \leq \varepsilon$.

Then $|a_n - A||b_n - B| + |a_n - A|B + A|b_n - B| \leq \varepsilon_1^2 + \varepsilon_1(A+B) \leq \varepsilon$.

, $\forall n \geq N_1$

This shows $\lim_{n\to\infty} a_n b_n = AB$.

Similarly, one can show other results in this theorem. ▨

Thm. Let $F: \mathbb{R} \to \mathbb{R}$ be a continuous function.

Then, if $\lim_{n\to\infty} a_n = A$, then $\lim_{n\to\infty} F(a_n) = F(A)$.

(proof) Fix arbitrary $\varepsilon > 0$.

Since F is continuous,

$\exists \delta > 0$ s.t. $|F(x) - F(A)| < \varepsilon$ for all $|x - A| < \delta$.

- Now, choose N s.t. $|a_n - A| \leq \delta$. ← possible since $\lim_{n\to\infty} a_n = A$.

for all $n \geq N$

- Therefore, for all $n \geq N$, $|F(a_n) - F(A)| \leq \varepsilon$.

This proves $\lim_{n\to\infty} F(a_n) = F(A)$. ▨

Thm Suppose $\{a_n\}$ is monotonically increasing. i.e. $a_n \le a_{n+1} \le a_{n+2} \le \cdots$

(resp, decreasing $a_n \ge a_{n+1} \ge a_{n+2} \ge \cdots$)

for all $n \ge N$.

Also, assume that $a_n \le M$ (resp, $a_n \ge M$) for all $n \ge N$.

※ Here, M & N are fixed constants.

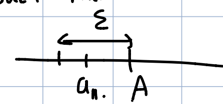
Then $\lim_{n\to\infty} a_n$ exists & $\lim_{n\to\infty} a_n \le M$ (resp, $\lim_{n\to\infty} a_n \ge M$).

(proof). In the mon. increasing case.

Let $\boxed{A = \sup_{n \ge N} a_n}$ (In the decreasing case can take $A = \inf_{n\ge N} a_n$).

Then, $\forall \varepsilon > 0$, there exists $n_1 \ge N$ such that

$$|a_{n_1} - A| \le \varepsilon.$$



Now $\forall n \ge n_1$, $a_{n_1} \le a_n \le A$

thus, $|a_n - A| \le \varepsilon$.

This shows the convergence. ▨

Existence of such "supremum" follows from "completeness" axiom of real numbers.

Rmk $\{a_n\}$ increasing

- $\lim_{n\to\infty} a_n = L$ $L \in \mathbb{R}$.
- $\lim_{n\to\infty} a_n = +\infty$ ← It means $\forall M$, there exists N s.t. $\forall n \ge N$, $a_n \ge M$.
  (i.e. a_n can be larger than any number for large enough n)

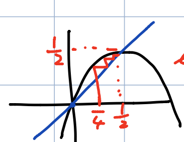
EX $F(x) = 2x(1-x)$.

$a_1 = \frac{1}{4}$, $a_2 = F(a_1)$, $a_3 = F(a_2)$, $\cdots$ $a_n = F(a_{n-1})$.

Does this sequence converge? If so, find its limit.

<sol>.

- Guess:



See geometrically, $\underbrace{a_n \text{ increasing}}_{①}$ and $\underbrace{\text{bounded by } \tfrac{1}{2}}_{②}$.

- Let us conform ① & ② analytically.

$F(x) = 2x(1-x) = 2x - 2x^2$

- For $0 \le x \le \frac{1}{2}$, $1 - x \ge \frac{1}{2}$, so, $2x(1-x) \ge 2x \cdot \frac{1}{2} \ge x$. so. $\underline{F(x) \ge x}$ on $0 \le x \le \frac{1}{2}$.

- " , $F'(x) = 2 - 4x \ge 0$, so $F(x) \le F(\frac{1}{2}) = \frac{1}{2}$. $\therefore$ $\underline{F(x) \le \frac{1}{2}}$ on $0 \le x \le \frac{1}{2}$.

From these, since $a_1 = \frac{1}{4} \le \frac{1}{2}$, this shows $\{a_n\}$ is increasing & $a_n \le \frac{1}{2}$ $\forall n \ge 1$.

So, the limit exists.

Let $\lim_{n\to\infty} a_n = L$.

Then $\lim_{n\to\infty} F(a_n) = \lim_{n\to\infty} a_{n+1} = L$ $\Rightarrow F(L) = L$

$F(x) = 2x(1-x)$ is continuous $\longrightarrow$ $\| \quad F(\lim_{n\to\infty} a_n)$ $\Rightarrow 2L(1-L) = L$

$= F(L)$ $\Rightarrow \underline{\underline{L = \frac{1}{2}}}$.

$\therefore \lim_{n\to\infty} a_n = \frac{1}{2}$. $\square$

- Without monotonicity (increasing/decreasing) /or without explicit formula $a_n = f(n)$, it can be subtle to show existence of limit.

In the Next lectures, we will see some interesting examples regarding iterated maps.