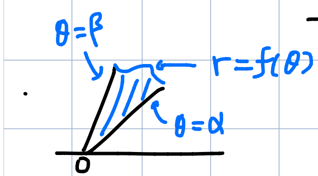
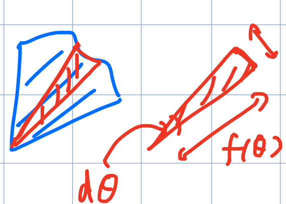
Lec32. Polar curves.

- area
- arc-length. §8.6.





$dA = \frac{1}{2} f(\theta) \cdot f(\theta) d\theta$

$$\therefore \text{Area} = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} |f(\theta)|^2 d\theta$$

e.g.

$r = \sin(2\theta)$
$0 \le \theta \le \frac{\pi}{2}$.

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} (\sin(2\theta))^2 d\theta$$

$\leftarrow \sin^2 u = \frac{1-\cos 2u}{2}$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1-\cos(4\theta)}{2} d\theta$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{2}} [1 - \cos(4\theta)] d\theta$$

$$= \frac{1}{4} \left[\theta - \frac{\sin(4\theta)}{4} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8}$$ ▨

e.g. Find the area lying inside $r = a$ & $\underbrace{r = 2a\cos\theta}$ $(a>0)$.

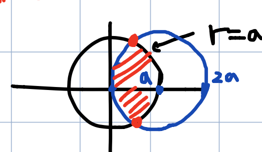
↑ a circle.

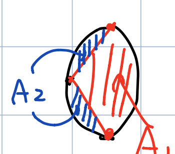
$r^2 = 2ar\cos\theta$

$x^2 + y^2 = 2ax$.

※ The solution demonstrates a strategy to handle general polar curves.

Sketch.



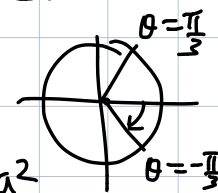


$A = A_1 + A_2$

Intersections $\{r=a\} \cap \{r=2a\cos\theta\}$

$$a = 2a\cos\theta \implies \cos\theta = \frac{1}{2}$$

$$\implies \theta = \frac{\pi}{3}, -\frac{\pi}{3}. \quad \text{for } -\pi \le \theta \le \pi$$



$$\therefore A_1 = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2} a^2 \, d\theta = \frac{1}{2}a^2 \cdot 2 \cdot \frac{\pi}{3} = \frac{\pi a^2}{3}.$$

due to symmetry

$$A_2 = 2 \cdot \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2} \cdot (2a\cos\theta)^2 d\theta = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2a^2\cos^2\theta \, d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} a^2 \cdot \frac{1+\cos 2\theta}{2} d\theta$$

$$= a^2\left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

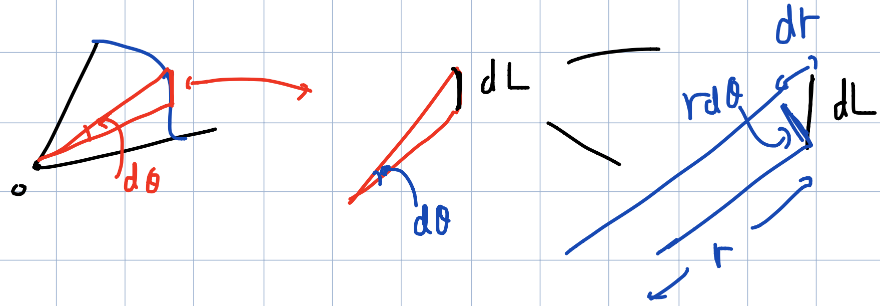
$$= 2 \cdot a^2 \left(\frac{\pi}{4} + \cancel{\frac{\sin(\pi)}{4}}^{\,0} - \left(\frac{\pi}{6} + \frac{\sin\frac{2\pi}{3}}{4}\right)\right]$$

$$= 2a^2\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right)$$

$$\therefore A = A_1 + A_2 = \frac{\pi a^2}{3} + 2a^2\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right)$$

$$= \underline{a^2\left[\frac{\pi}{2} - \frac{\sqrt{3}}{4}\right]}. \quad \blacksquare$$

Arc-length.



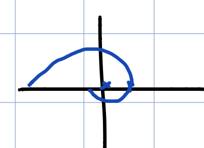
$$dL^2 = (r\,d\theta)^2 + (dr)^2$$

$$= \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right] d\theta$$

$$\therefore\ dL = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad r = f(\theta)$$

$$\therefore\ \boxed{dL(\theta) = \sqrt{f^2(\theta) + (f'(\theta))^2}\, d\theta.}$$

EX arc-length $r = e^{a\theta}$. $-\pi \le \theta \le \pi$



$$L = \int_{\theta=-\pi}^{\theta=\pi} dL = \int_{-\pi}^{\pi} \sqrt{(e^{a\theta})^2 + (a e^{a\theta})^2}\, d\theta$$

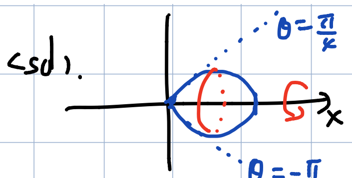
$$= \int_{-\pi}^{\pi} e^{a\theta}\sqrt{1+a^2}\ d\theta$$

$$= \sqrt{1+a^2}\left[\frac{e^{a\theta}}{a}\right]_{-\pi}^{\pi} = \underline{\frac{\sqrt{1+a^2}}{a}\left[e^{\pi a} - e^{-\pi a}\right]}.$$

EX. $r^2 = \cos(2\theta)$.

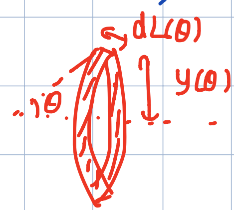
Find the surface area of the surface generated by rotating one leaf of the curve about x-axis

<sol1>.



Surface area

$$S = \int_{\theta=0}^{\theta=\frac{\pi}{4}} 2\pi\, y(\theta)\cdot dL(\theta)$$



$$= \int_0^{\frac{\pi}{4}} 2\pi \cdot r(\theta)\sin\theta \sqrt{[r(\theta)]^2+[r'(\theta)]^2}\,d\theta$$

$y(\theta)=r(\theta)\sin\theta$

$$= \int_0^{\frac{\pi}{4}} 2\pi\sin\theta \sqrt{[r(\theta)]^4+[r(\theta)\,r'(\theta)]^2}\,d\theta$$

$$r^2 = \cos(2\theta)$$

$$2r\frac{dr}{d\theta} = -2\sin(2\theta)$$

$$r\frac{dr}{d\theta} = -\sin(2\theta)$$

$$= \int_0^{\frac{\pi}{4}} 2\pi\sin\theta \sqrt{\cos^2(2\theta)+\sin^2(2\theta)}\,d\theta$$

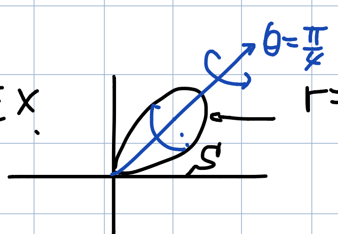
$$= \int_0^{\frac{\pi}{4}} 2\pi\sin\theta\,d\theta$$

$$= \Big[-2\pi\cos\theta\Big]_0^{\frac{\pi}{4}}$$

$$= 2\pi\left[-\frac{\sqrt{2}}{2}+1\right]$$

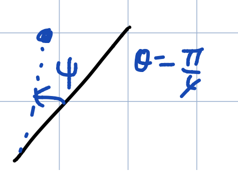
$$= 2\pi\left(1-\frac{\sqrt{2}}{2}\right).$$ ▨

Ex.



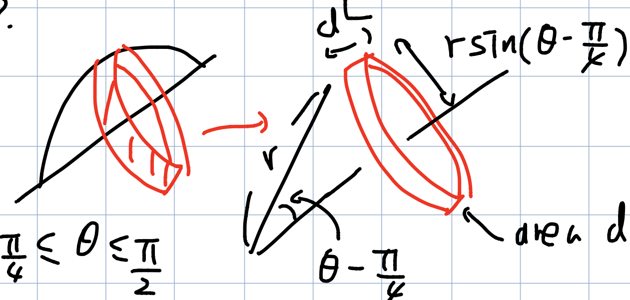
$r=\sin 2\theta$, S = the surface is generated by rotating $r=\sin 2\theta$, $0\le\theta\le\frac{\pi}{2}$ about the line $r=\frac{\pi}{4}$.

Mass density: $\rho=\frac{1}{\sin\psi}$.
per surface area



Find the total mass.

<sd>.



$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

area $dS(\theta) = 2\pi\, r(\theta)\sin(\theta-\frac{\pi}{4})\ dL(\theta)$.

$$\therefore\ dS(\theta) = 2\pi\, r(\theta)\sin(\theta-\tfrac{\pi}{4})\sqrt{r^2(\theta)+[r'(\theta)]^2}\ d\theta$$

$r(\theta) = \sin(2\theta)$

$r'(\theta) = 2\cos(2\theta)$ $\longrightarrow$ $= 2\pi\sin(2\theta)\sin(\theta-\frac{\pi}{4}) \times \sqrt{1+3\cos^2(2\theta)}\ d\theta$

$\therefore\ r(\theta)^2+[r'(\theta)]^2$

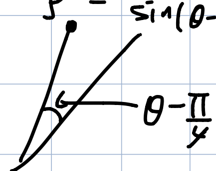
$= \sin^2(2\theta) + 4\cos^2(2\theta)$

$= 1 + 3\cos^2(2\theta)$.

Mass $\quad dM(\theta) = \rho \cdot dS(\theta)$

$$= \frac{1}{\sin(\theta-\frac{\pi}{4})}\cdot 2\pi\sin(2\theta)\cdot\sin(\theta-\tfrac{\pi}{4})\sqrt{1+3\cos^2(2\theta)}\,d\theta$$

$\rho = \frac{1}{\sin(\theta-\frac{\pi}{4})}$



$$= 2\pi\ \sin(2\theta)\sqrt{1+3\cos^2(2\theta)}\ d\theta.$$

$\therefore$ Total mass

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\pi\ \sin(2\theta)\sqrt{1+3\cos^2(2\theta)}\ d\theta$$

$u = \cos 2\theta$

$du = -\sin 2\theta\, d\theta$

$$= 2\pi\int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \sqrt{1+3u^2}\cdot(-du)$$

$$= -2\pi \int_0^{-1} \sqrt{1+3u^2}\, du$$

$u = \frac{1}{\sqrt{3}} \tan\alpha$,

$du = \frac{1}{\sqrt{3}} \sec^2\alpha \, d\alpha$

$$= 2\pi \int_{-1}^{0} \sqrt{1+3u^2}\, du$$

symmetry

$$= 2\pi \int_0^1 \sqrt{1+3u^2}\, du$$

$1+3u^2 = 1+\tan^2\alpha$

$= \sec^2\alpha$.

$$= 2\pi \int_0^{\frac{\pi}{3}} |\sec\alpha| \frac{\sec^2\alpha}{\sqrt{3}}\, d\alpha$$

$$= \frac{2\pi}{\sqrt{3}} \int_0^{\frac{\pi}{3}} \sec^3\alpha \, d\alpha \quad \cdots\cdots (*)$$

Let us compute.

$$I = \int \underbrace{\sec\alpha}_{u} \, \underbrace{\sec^2\alpha \, d\alpha}_{dv}$$

Integration by parts

$$= \underset{u}{\sec\alpha} \cdot \underset{v}{\tan\alpha} - \int \underset{v}{\tan\alpha} \cdot \underset{du}{\sec\alpha\tan\alpha\, d\alpha}$$

$$= \sec\alpha\tan\alpha - \int \sec\alpha\,(\sec^2\alpha - 1)\, d\alpha$$

$$= \sec\alpha\tan\alpha + \int \sec\alpha\, d\alpha - \underbrace{\int \sec^3\alpha\, d\alpha}_{I}.$$

$$\therefore \quad 2I = \sec\alpha\tan\alpha + \int \sec\alpha\, d\alpha$$

$$= \sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha| + C$$

$$\therefore \quad I = \frac{1}{2}\left[\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha| + C\right]$$

$$\left[I \right]_0^{\frac{\pi}{3}} = \frac{1}{2}\left[\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\Big[\sec\frac{\pi}{3}\tan\frac{\pi}{3} + \ln\Big|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\Big|$$

$$- \underbrace{\sec 0 \cdot \cancel{\tan 0}}_{0} - \underbrace{\ln|\underbrace{\sec 0}_{=1} + \underbrace{\cancel{\tan 0}}_{0}|}_{0} \Big]$$

$$= \frac{1}{2}\Big[2 \cdot \frac{\sqrt{3}}{2} + \ln\Big| 2 + \frac{\sqrt{3}}{2}\Big| \Big]$$

Plug-in this to ⊛,

and get the total mass

$$\frac{2\pi}{\sqrt{3}} \cdot \frac{1}{2}\Big[\sqrt{3} + \ln\Big|2 + \frac{\sqrt{3}}{2}\Big| \Big]$$

$$= \frac{\pi}{\sqrt{3}}\Big[\sqrt{3} + \ln\Big|2 + \frac{\sqrt{3}}{2}\Big|\Big] \quad ▨$$