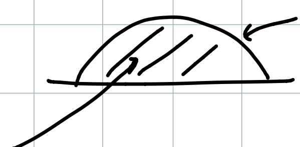
Lec 30. Parametric curves.

· Area bounded by parametric curves §8.4.

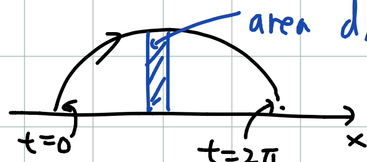
cycloid

Ex. 

$x = a(t - \sin t)$

$y = a(1 - \cos t)$.

Area ?

<sol>. 

area $dA = y\,dx$.

$$\text{Area } A = \int_{t=0}^{t=2\pi} dA = \int_{t=0}^{t=2\pi} y\,dx$$

$$= \int_0^{2\pi} a(1-\cos t)\cdot \underbrace{a(1-\cos t)\,dt}_{dx(t) = (a(t-\sin t))'\,dt}$$

$$= a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt$$

$\cos 2t = \cos^2 t - \sin^2 t$

$= 2\cos^2 t - 1$

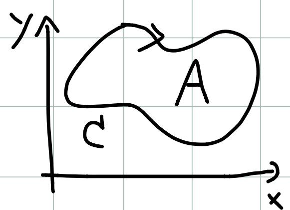
$\therefore \cos^2 t = \frac{1+\cos 2t}{2}$

$$= a^2 \Big[t - 2\sin t + \underbrace{\frac{t}{2} + \frac{\sin 2t}{4}}_{\int \cos^2 t\, dt} \Big]_0^{2\pi}$$

$$= a^2 \cdot 3\pi$$

$$= \underline{3\pi a^2}.$$

Area of the region bounded by a closed curve:



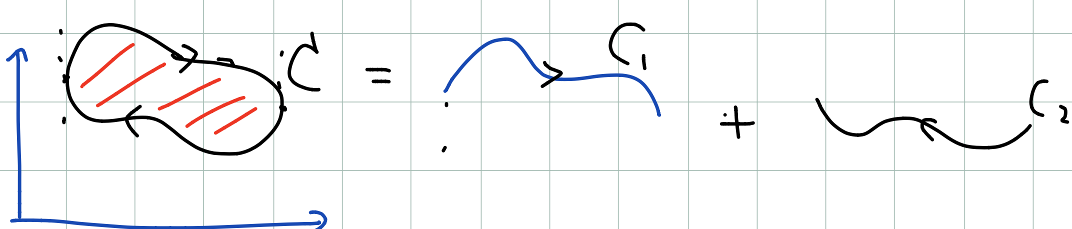
C is oriented clockwise

$$\text{Area} = \int_{\overset{\curvearrowright}{C}} y\,dx$$

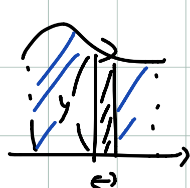
"notation for the integral along the oriented curve C".

※ In multivariable calculus, you will learn Green's theorem that explains this in an elegant way.

Explanation



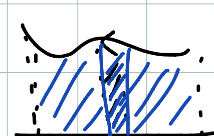
$\int_{C_1} y\,dx$ = area of



along C_1 (in the given direction).

$dx > 0$.

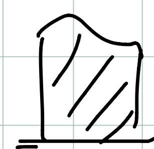
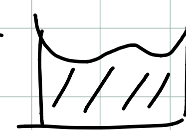
$\int_{C_2} y\,\underbrace{dx}$ = −area of



small change in x as moving along C_2 in the given direction

Note along C_2, $dx < 0$.

$$\therefore \int_C y\,dx = \int_{C_1} y\,dx + \int_{C_2} y\,dx$$

= area of  $-$ area of 

= area of .

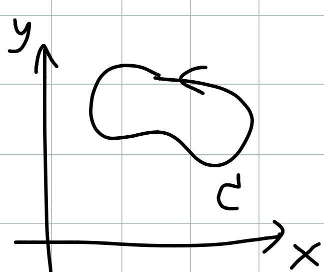
$\blacksquare$

Similarly, if C' is oriented <u>counter clockwise</u>

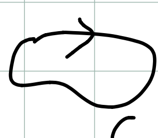
Then

area $= -\int_C y\,dx$.

<u>Note</u> Using the same idea can show



Area $= \int_{C'} x\,dy$ in the counter-clockwise case



Area $= -\int_C x\,dy$ in the clockwise case.

Rmk

$$A = \frac{1}{2} \int_C (y\,dx - x\,dy) \quad \text{in clockwise case}$$

$$A = \frac{1}{2} \int_C (x\,dy - y\,dx) \quad \text{in counterclockwise case.}$$

Also $$\int_C (x\,dy + y\,dx) = 0$$

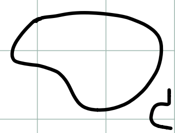
Rmk

These are special cases of Green's theorem.

"The idea of Green's theorem is to relate integrals on a domain (like area) with integrals along its boundary".

You will see this in vector calculus courses.

On parametric curves



$$x = x(t), \quad y = y(t)$$

suppose the curve is closed for $a \le t \le b$

i.e. $x(a) = x(b)$
$y(a) = y(b)$.

Then the area bounded by the curve

clockwise case :

$$A = \int_{t=a}^{t=b} y \, dx = \int_a^b y(t)\, x'(t)\, dt$$

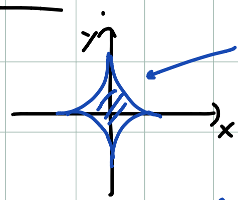
$$A = -\int_{t=a}^{t=b} x \, dy = -\int_a^b x(t)\, y'(t)\, dt$$

counterclockwise case :

$$A = \int_{t=a}^{t=b} x \, dy = \int_a^b x(t)\, y'(t)\, dt$$

$$A = -\int_{t=a}^{t=b} y \, dx = -\int_a^b y(t)\, x'(t)\, dt$$

EX



$x = a\cos^3 t$

$y = a\sin^3 t$. $0 \le t \le 2\pi$

Area = ?

<sol>. Note the parametric curve moves Counterclockwise

$$\text{The area} = \int_{t=0}^{t=2\pi} x\,dy$$

$$= \int_0^{2\pi} a\cos^3 t \cdot a \cdot 3\sin^2 t\cos t\,dt$$

$$= 3a^2 \int_0^{2\pi} \cos^4 t\,\sin^2 t\,dt$$

$$= 3a^2 \int_0^{2\pi} \cos^4 t \cdot (1-\cos^2 t)\,dt$$

$$= 3a^2 \left[\int_0^{2\pi} \cos^4 t\,dt - \int_0^{2\pi} \cos^6 t\,dt \right]$$

$$I_n = \int_0^{2\pi} \cos^{2n} x\,dx$$

$$= \int_0^{2\pi} \cos x \ \cos^{2n-1} x\,dx$$

$$= \sin x \cdot \cos^{2n-1} x \Big]_0^{2\pi} - \int_0^{2\pi} \sin x \cdot (2n-1)\cos^{2n-2} x \cdot (-\sin x)\,dx$$

$$= 0 + (2n-1)\int_0^{2\pi} \cos^{2n-2} \cdot \sin^2 x\,dx$$

$$= (2n-1)\int_0^{2\pi} \left(\cos^{2n-2} x - \cos^{2n} x\right) dx \quad \leftarrow \sin^2 x = 1 - \cos^2 x$$

$$= (2n-1) I_{(n-1)} - (2n-1) I_n$$

$$\therefore \quad 2n I_n = (2n-1) I_{n-1}$$

$$6 I_3 = 3 \cdot I_2 \,, \qquad 4 I_2 = I_0 = 2\pi$$

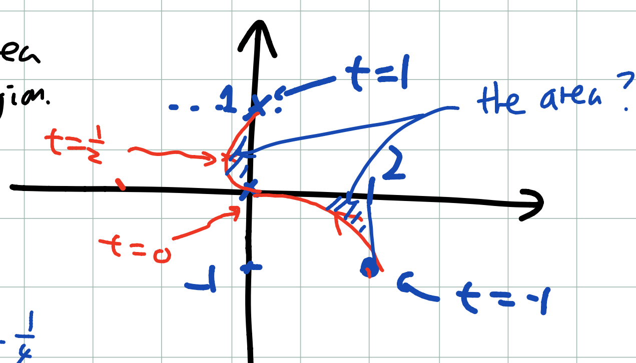
$$\therefore \quad I_2 = \frac{\pi}{2} \,. \quad I_3 = \frac{\pi}{4}$$

$\therefore$ The area $= 3a^2 \left[I_2 - I_3 \right]$

$$= 3a^2 \left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \underline{\frac{3\pi}{4} a^2}$$

Ex. $x = t^2 - t \qquad y = t^3 . \qquad -1 \le t \le 1 .$

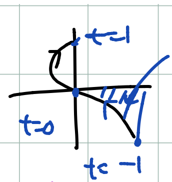
Find the area of the shaded region.



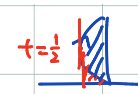
$x(\frac{1}{2}) = -\frac{1}{4}$

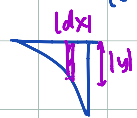
$y(\frac{1}{2}) = \frac{1}{8}$

< sol >.



$$\text{Area} = \int_{t=-1}^{t=0} y\,dx + \int_{t=0}^{t=1} y\,dx$$





$dA = |y||dx|$

$= y\,dx$ since $y <$, $dx<0$

along

Area

$=$  $-$ 

$$= \int_{t=\frac{1}{2}}^{t=1} y\,dx - \int_{t=\frac{1}{2}}^{t=0} y\,dx$$

$$= \int_{t=0}^{t=\frac{1}{2}} y\,dx.$$

$$\therefore \text{area} = \int_{t=-1}^{t=1} y\,dx$$

$$= \int_{-1}^{1} t^3(2t-1)\,dt \qquad x = t^2 - t,\quad dx = (2t-1)\,dt$$

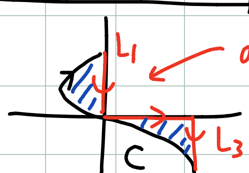
$$= \int_{-1}^{1} 2t^4 - t^3\,dt \qquad -t^3 \text{ odd. on } [-1,1]$$

$$= 2\int_{-1}^{1} t^4\,dt - \cancel{\int_{-1}^{1} t^3\,dt},\ 0$$

$$= 4\cdot\int_0^1 t^4\,dt$$

$$= 4\cdot\left[\frac{t^5}{5}\right]_0^1 = \frac{4}{5} \quad \boxed{/\!/}$$

Another view point.



add L_1, L_2, L_3 to C to make a closed curve $\hat{C}$.

Then the area

$$A = \int_{\hat{C}} y\,dx \leftarrow \hat{C} \text{ clockwise}$$

But

$$\int_{\hat{C}} y\,dx = \int_C y\,dx + \cancel{\int_{L_1} y\,dx}\ 0 + \cancel{\int_{L_2} y\,dx}\ 0 + \cancel{\int_{L_3} y\,dx}\ 0$$

$dx=0$ along L_1 $\qquad$ $y=0$ along L_2 $\qquad$ $dx=0$ along L_3.

same as before

$$= \int_C y\,dx = \int_{t=-1}^{t=1} y(t)\,dx(t) = \int_{t=-1}^{t=1} t^3(2t-1)\,dt = \frac{4}{5} \quad \boxed{/\!/}$$