Lec 29. · parametric curves
- sketching §8.3.
- arc-length, area §8.4

• Shape of a parametric curve: Concavity / convexity

For $x = f(t),\ y = g(t).$ $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$ for $f'(t) \neq 0$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right] \Big/ \frac{dx}{dt}$$

$$= \left[\frac{g'(t)}{f'(t)}\right]' \Big/ f'(t) = \frac{g''(t)f'(t) - g'(t)f''(t)}{[f'(t)]^2} \Big/ f'(t)$$

$$= \frac{1}{[f'(t)]^3}\left[g''(t)f'(t) - g'(t)f''(t)\right] \quad \text{for} \quad f'(t) \neq 0.$$

• sketching a parametric curve using second derivatives.

Ex. $x = t^2 - t \qquad y = t^3. \qquad -1 \leq t \leq 1.$

$f(t) = t^2 - t \qquad f'(t) = 2t - 1$.

$g(t) = t^3 \qquad g'(t) = 3t^2.$

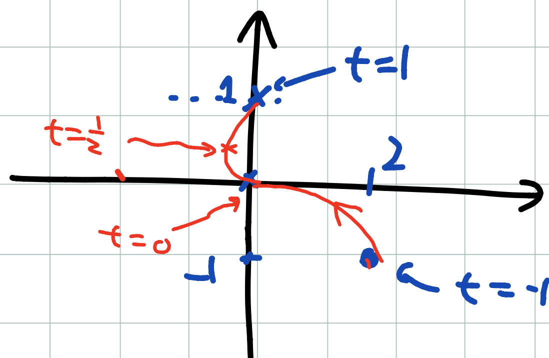
t		0		$\frac{1}{2}$	
$f'(t)$		$-$		0	$+$
$g'(t)$	$+$	0		$+$	
x		$\leftarrow$		$\cdot$	$\rightarrow$
y	$\uparrow$	$\cdot$		$\uparrow$	
curve	$\nwarrow$	$\leftarrow$	$\nwarrow$	$\uparrow$	$\nearrow$

For concavity
$f''(t) = 2$
$g''(t) = 6t$

$$\frac{d^2y}{dx^2} = \frac{1}{(f')^3}\left[g''f' - g'f''\right] = \frac{1}{(2t-1)^3}\left[6t(2t-1) - 3t^2 \cdot 2\right]$$

$$= \frac{6t(t-1)}{(2t-1)^3}$$

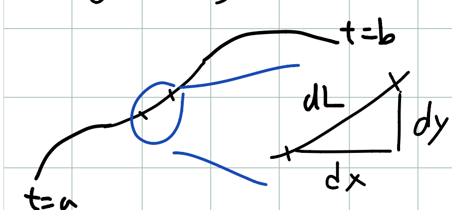
t		0		$\frac{1}{2}$		1	
$\frac{d^2y}{dx^2}$	$-$	0	$+$	$\pm\infty$	$-$	0	$+$
curve	concave down		concave down		concave down		concave up



$x(\frac{1}{2}) = -\frac{1}{4}$

$y(\frac{1}{2}) = \frac{1}{8}$

- Arc-length §8.4



$x = f(t)$ $\quad a \le t \le b$

$y = g(t)$

$$dL = \sqrt{dx^2 + dy^2}$$

In terms of t, $dL = \sqrt{\left(\frac{dx}{dt}dt\right)^2 + \left(\frac{dy}{dt}dt\right)^2}$

$$= \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]dt^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

arc-length

$$L = \int_{t=a}^{t=b} dL = \int_{t=a}^{t=b} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

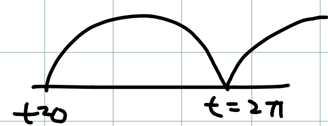
Typically, calculating this integral is difficult.

Some simple case:

Ex Arc-length of a cycloid

$x = f(t) = a(t - \sin t)$

$y = g(t) = a(1 - \cos t)$



Arc-length for $0 \le t \le 2\pi$

$$L = \int_{t=0}^{t=2\pi} \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

$f'(t) = a(1 - \cos t)$

$g'(t) = a \sin t$

$$= \int_0^{2\pi} a\sqrt{2 - 2\cos t}\, dt$$

$\therefore\ [f']^2 + [g']^2$

$= a^2(1 - 2\cos t + \cos^2 t + \sin^2 t)$

$= a^2(2 - 2\cos t)$

$$= \int_0^{2\pi} a\sqrt{4\sin^2 \tfrac{t}{2}}\, dt = \int_0^{2\pi} \underbrace{2a\sqrt{\sin^2 \tfrac{t}{2}}}_{dL(t)}\, dt$$

$\cos t = \cos^2 \frac{t}{2} - \sin^2 \frac{t}{2}$

$\cos t = 1 - 2\sin^2 \frac{t}{2}$

$2\sin^2 \frac{t}{2} = 1 - \cos t.$

$$= a \int_{u=0}^{u=\pi} 4\sqrt{\sin^2 u}\, du$$

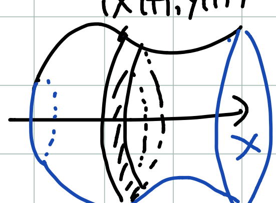
$$= 4a \int_0^{\pi} \sin u\, du$$

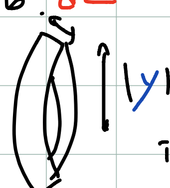
$$= 4a\, [-\cos u]_0^{\pi}$$

$$= 4 \cdot 2a = 8a. \quad \boxed{/\!/}$$

• Area of Surface of revolution

$(x(t), y(t))$ $a \le t \le b$. dL





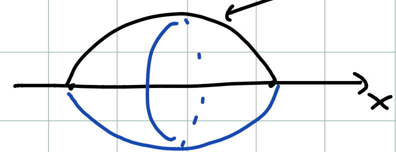
$dS = 2\pi |y| dL$

in terms of t

$$= 2\pi |y(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$S = \int_{t=a}^{t=b} ds = \int_a^b 2\pi\,|y(t)|\,\underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}_{dL}$$

Ex. cycloid



$x = a(t - \sin t)$

$y = a(1 - \cos t)$

$0 \le t \le 2\pi$

The surface area = ?

<sol>.

$$S = \int_{t=0}^{t=2\pi} 2\pi \cdot |y(t)|\, dL(t)$$

$$= \int_0^{2\pi} 2\pi\, a(\underbrace{1-\cos t}_{2\sin^2\frac{t}{2}}) \cdot 2a\sqrt{\sin^2\frac{t}{2}}\, dt$$

$$= 8\pi a^2 \int_0^{2\pi} \sin^2\frac{t}{2} \cdot \sin\frac{t}{2}\, dt$$

← note $\sin\frac{t}{2} \ge 0$ for $0 \le t \le 2\pi$

$$= 8\pi a^2 \int_0^{\pi} \sin^2 u \cdot \sin u \cdot 2\,du$$

$u = \frac{t}{2}$

$2du = dt$.

$$= 16\pi a^2 \int_0^{\pi} (1 - \cos^2 u) \sin u\, du$$

$$= 16\pi a^2 \int_1^{-1} (1 - w^2) \cdot (-dw)$$

$w = \cos u$

$dw = -\sin u\, du$

$$= 16\pi a^2 \int_{-1}^{1} (1 - w^2)\, dw = 16\pi a^2 \cdot 2\int_0^1 (1 - w^2)\, dw$$

$$= 32\pi a^2 \left[w - \frac{w^3}{3} \right]_0^1 = 32\pi a^2 \left[1 - \frac{1}{3} \right]$$

$$= \frac{64\pi a^2}{3}.$$

$\boxtimes$