Lec 26

§7.8. Probability.

- Last lecture
  - probability density
  - Cumulative function
  - mean (= expected value)

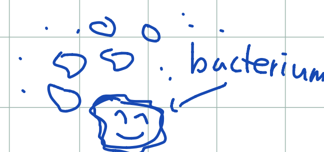
- Today: { - Change of variable; - Zeroth, First, second momenta.

(Do slide lecture).

- **Change of variable** in probability:

- From probability density of x to probability density of $u = f(x)$.

EX A bacteria colony



- Suppose for each bacterium its life time t (hrs) depends on its volume, v (cubic micro-meter) given as $t = e^v$.

Suppose the volume of a bacterium has probability density function for v,

$$p(v) = \frac{1}{5000} v \qquad 0 \le v \le 100$$

(a) Find the probability density function for t.

(b) Find the expected life time $\bar{t}$.

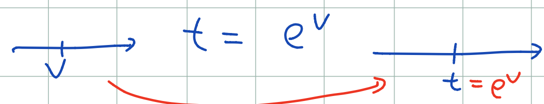
<sol>

(a) ∴ We know $p(v)$.

- Want to know the probability density $q(t)$ for t.

(We used different letter q to avoid confusion with $p(v)$).

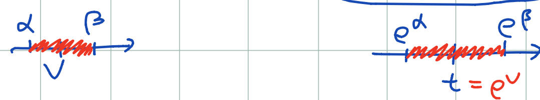
- Relation between t & v



- How to translate $p(v)$ to $q(t)$?

Important observation

The group of bacteria having v within $[\alpha, \beta]$	=	The group of bacteria having $t = e^v$ within $[e^\alpha, e^\beta]$

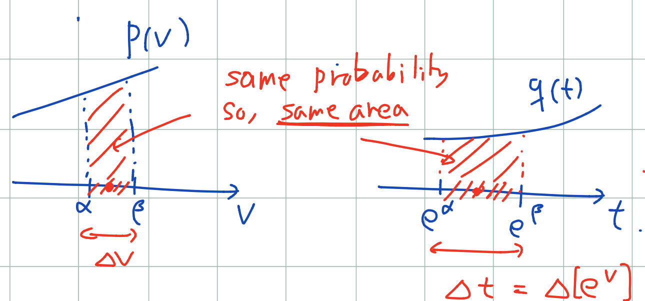


So, the <u>same portion</u> of bacteria corresponds to <u>both</u>.

This means

$$\left(\begin{array}{l}\text{Probability}\\ \text{for } \alpha \le v \le \beta\end{array}\right) = \left(\begin{array}{l}\text{Probability}\\ \text{for } e^\alpha \le t \le e^\beta\end{array}\right).$$

So,



So, <u>approximately</u>,

$$p(v)\,\Delta v \approx q(t)\,\Delta t \quad \text{for } t = e^v.$$

<u>Infinitesimally</u> (i.e. as $\Delta v \to 0$),

$$\boxed{p(v)\,dv = q(t)\,dt.}$$

The key formula in this discussion

- Now, Use $\begin{cases} t = e^v \\ dt = e^v dv \end{cases}$

$$p(v)\,dv = q(t)\,e^v dv$$

$\therefore\ q(t) = e^{-v} p(v)$ use as given,

$= e^{-v} \cdot \frac{v}{5000} \leftarrow p(v) = \frac{1}{5000} v$

- In terms of t, $t = e^v$, $\therefore v = \ln t$.

$\cdot\ e^{-v} = \frac{1}{t}$.

So, $q(t) = \frac{1}{5000t} \ln t$.

- Determine the domain of $q(t)$.

(= possible range for the random variable t).

possible range for v

$0 \le v \le 100 \implies 1 = e^0 \le t \le e^{100}$. $1 \le t \le e^{100}$

possible range for t.

- Conclusion:

The probability density for t,

$$q(t) = \frac{1}{5000t} \ln t, \quad 1 \le t \le e^{100}.$$

$\blacksquare$ (a).

(b): expected value $\bar{t} = \int_1^{e^{100}} t\, q(t)\, dt$

- One can directly compute this integral using $q(t)$ from (a).

OR

- Can use the relation $\begin{cases} t = e^v, \\ q(t)\,dt = p(v)\,dv \end{cases} \leftarrow$ from the solution of (a).

$$\bar{t} = \int_1^{e^{100}} t\, q(t)\, dt$$

$$= \int_{v=0}^{v=100} e^v\, p(v)\, dv.$$

$$= \int_{v=0}^{v=100} e^v \cdot \frac{v}{5000}\, dv$$

$$= \frac{1}{5000}\int_0^{100} v e^v\, dv$$

integration by parts

$df = e^v dv$

$g = v$.

$$= \frac{1}{5000}\left\{ \left[v e^v \right]_0^{100} - \int_0^{100} 1 \cdot e^v\, dv \right\}$$

$$= \frac{1}{5000}\left\{ 100 e^{100} - 0 - e^{100} + e^0 \right\}$$

$$\therefore \bar{t} = \frac{1}{5000}\left\{ 99 e^{100} + 1 \right\} \qquad \blacksquare \text{ (b)}$$

Rmk In the above example

- the expected value of volume

$$\bar{V} = \int_0^{100} v \cdot p(v)\, dv$$

$$= \int_0^{100} v \cdot \frac{v}{5000}\, dv$$

$$= \frac{1}{5000}\left[\frac{1}{3} v^3 \right]_0^{100}$$

$$= \frac{1}{5000} \cdot \frac{1}{3} \cdot 100^3 = \frac{1000}{5 \cdot 3} = \frac{200}{3}.$$

Note $t = e^v$.

So, the expected life-time

$$\cancel{\bar{t} = e^{\bar{V}} = e^{\frac{200}{3}}}$$

WRONG!

NOTE WARNING!

For v random variable

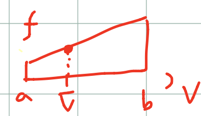
$t = f(v)$ $\quad \bar{t}$ = mean of t.

$\bar{v}$ = mean of v.

Does NOT hold in general.

$$\cancel{\bar{t} = f(\bar{v})} \leftarrow \text{WRONG!}$$

$\leftarrow$ $\bar{t} = f(\bar{v})$ works if f is linear.



For strictly convex f,

$\bar{t} > f(\bar{v})$

Note Jensen's inequality

For a random variable x & a convex function f,

Expected value of $f(x) \geq f($expected value of $x)$.

§ moments & · mean · variance · standard deviation.

Def Let $f(x) \geq 0$. $a \leq x \leq b$.

k-th Moment: $M_i = \int_a^b x^k f(x)\,dx$.

i.e.

- zero'th moment: $M_0 = \int_a^b f(x)\,dx$
- first moment: $M_1 = \int_a^b x f(x)\,dx$
- second moment: $M_2 = \int_a^b x^2 f(x)\,dx$

Moments are defined for nonnegative functions that are not necessarily probability densities.

For a probability density function $p(x)$, $a \leq x \leq b$ $\leftarrow$ domain of the random variable x.

- zeroth moment $M_0 = \int_a^b p(x)\,dx = 1$.
- first moment $M = \int_a^b x\,p(x)\,dx = \bar{x}$ $\leftarrow$ the mean.

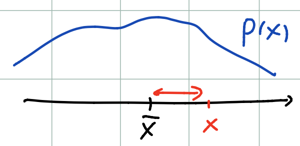
- Second moment $M_2 = \int_a^b x^2 \, p(x)\, dx$

the second moment
for a probability measure is related to

the **variance** of the random variable x

$$V = \int_a^b (x - \bar{x})^2 \, p(x)\, dx$$

$\bar{x}$ = mean of x
(= average of x)



The variance is the average distance squared
of the random variable x
from its mean $\bar{x}$.

Def Standard deviation

$$\sigma = \sqrt{V}.$$

Variance & second moment.

$$\boxed{V = M_2 - \bar{x}^2}$$

x random variable, $a \le x \le b$
$\bar{x}$ = the mean
V = the variance
M_2 = the second moment

This formula makes computation of variance a little bit simpler than using the definition

Reason for this formula:

$$V = \int_a^b (x - \bar{x})^2 \, p(x)\, dx$$

$$= \int_a^b \left[x^2 - 2x \cdot \bar{x} + \bar{x}^2\right] p(x)\, dx$$

$\bar{x}$ is a constant

$$= \underbrace{\int_a^b x^2 \, p(x)\, dx}_{= M_2} - 2 \int_a^b x \cdot \boxed{\bar{x}} \, p(x)\, dx$$

$$+ \int_a^b \boxed{\bar{x}}^2 \, p(x)\, dx$$

$$= M_2 - 2\bar{x} \underbrace{\int_a^b x\,p(x)\,dx}_{=\bar{x}} + \bar{x}^2 \underbrace{\int_a^b p(x)\,dx}_{=1}.$$

$$= M_2 - 2\bar{x} \cdot \bar{x} + \bar{x}^2$$

$$= M_2 - \bar{x}^2. \quad \boxtimes$$

EX. x random variable, $0 \le x \le 1$.

$p(x) = kx^2 + x$ probability density.

Find the variance & standard deviation.

<sol>. Recall the definition:

- Variance $V = \int_0^1 (x-\bar{x})^2\,p(x)\,dx$.

since $0 \le x \le 1$ in our case.

- So first need to determine

$p(x) \leftarrow$ need to determine the constant k.

$\bar{x} \leftarrow$ mean. $\bar{x} = \int_0^\infty x\,p(x)\,dx$.

- After determining these,
  - can compute the integral $V = \int_0^\infty (x-\bar{x})^2 p(x)\,dx$

or alternatively, one can use $\boxed{V = M_2 - \bar{x}^2}$.

simple to use.

- Determine $p(x)$. (i.e. the constant k).

$p(x) = kx^2 + x \quad 0 \le x \le 1$.

- To determine k,

Use total probability

$$1 = \int_0^1 (kx^2 + x)\,dx$$

So, $1 = \left[k\frac{x^3}{3} + \frac{x^2}{2} \right]_0^1$

$$= \frac{k}{3} + \frac{1}{2}$$

$\therefore \frac{1}{2} = \frac{k}{3}$

$\therefore k = \frac{3}{2}$

So $p(x) = \frac{3}{2}x^2 + x$

- Determine mean $\bar{x}$:

$$\bar{x} = \int^1 x\,p(x)\,dx = \int_0^1 x\left(\frac{3}{2}x^2 + x\right)dx$$

$$= \int_0^1 \left(\frac{3}{2}x^3 + x^2\right)dx$$

$$= \left[\frac{3}{2}\cdot\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1$$

$$= \frac{3}{2}\cdot\frac{1}{4} + \frac{1}{3}$$

$$= \frac{3}{8} + \frac{1}{3}$$

$$= \frac{3\cdot 3 + 1\cdot 8}{8\cdot 3} = \frac{9+8}{24}$$

$$= \frac{17}{24} \qquad \therefore \bar{x} = \frac{17}{24}.$$

- Let us use $V = M_2 - \bar{x}^2$

- Determine M_2:

$$M_2 = \int_0^1 x^2\,p(x)\,dx = \int_0^1 x^2\left(\frac{3}{2}x^2 + x\right)dx$$

$$= \int_0^1 \left(\frac{3}{2}x^4 + x^3\right) dx$$

$$= \left[\frac{3}{2}\cdot\frac{x^5}{5} + \frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{2}\cdot\frac{1}{5} + \frac{1}{4}$$

$$= \frac{3}{10} + \frac{1}{4}$$

$$= \frac{3\cdot 4 + 10}{4\cdot 10} = \frac{12+10}{40}$$

$$= \frac{22}{40}$$

$$= \underline{\underline{\frac{11}{20}}}. \qquad \underline{M_2 = \frac{11}{20}}$$

· Now use $V = M_2 - \bar{x}^2$

$$V = \frac{11}{20} - \left(\frac{17}{24}\right)^2$$

If time permits, we will do this long exercise.
(It involves improper integrals, L'Hopital, integration by parts.)

$\underline{EX}$. X random variable, $0 \leq x < \infty$.

$p(x) = e^{-kx}$ probability density.

($k>0$, a constant)

Find the variance & standard deviation.

<sol>. Recall the definition: improper integral.

- Variance $V = \int_0^\infty (x-\bar{x})^2 p(x)\, dx$.

since $0 \le x < \infty$ in our case.

- So first need to determine

$p(x) \leftarrow$ need to determine the constant k.

$\bar{x} \leftarrow$ mean. $\bar{x} = \int_0^\infty x\, p(x)\, dx$.

- After determining these,
  - can compute the integral $V = \int_0^\infty (x-\bar{x})^2 p(x) dx$

or alternatively, one can use $\boxed{V = M_2 - \bar{x}^2.}$

- Determine $p(x)$. (i.e. the constant k).

$$p(x) = e^{-kx}, \quad 0 \le x < \infty.$$

- To determine k,

Use total probability

$$1 = \int_0^\infty p(x)\, dx.$$

i.e. $1 = \int_0^\infty e^{-kx} dx = \boxed{\int_0^\infty} e^{-kx} dx$

improper integral

how to compute improper integral.

$$= \underline{\lim_{\alpha\to\infty} \int_0^\alpha} e^{-kx} dx$$

$$= \lim_{\alpha\to\infty} \left[-\frac{e^{-kx}}{k} \right]_0^\alpha$$

$$= \lim_{\alpha\to\infty} \left[-\frac{e^{-k\alpha}}{k} + \frac{1}{k} \right]$$

$$= \frac{1}{k} \leftarrow \lim_{\alpha\to\infty} e^{-k\alpha}$$

$= \lim_{\alpha \to \infty} \frac{1}{e^{k\alpha}} = 0.$

$k > 0$ so $k\alpha \to \infty$ as $\alpha \to \infty$.

So $\underline{k = 1}$.

So, $\underline{p(x) = e^{-x}}$

- Determine mean $\bar{x}$:

improper integral

$$\bar{x} = \int_0^\infty x\, p(x)\, dx = \int_0^\infty x e^{-x}\, dx$$

how to compute improper integral.

$$= \underbrace{\lim_{\alpha \to \infty}}\int_0^\alpha x e^{-x}\, dx$$

Integration by parts

$u = x$, $dv = e^{-x} dx$

Now,

$$\int_0^\alpha x e^{-x} dx = \left[x \cdot (-e^{-x})\right]_0^\alpha - \int_0^\alpha 1 \cdot (-e^{-x}) dx$$

$$= -\alpha e^{-\alpha} + 0 - \left[e^{-x}\right]_0^\alpha$$

$$= -\alpha e^{-\alpha} - e^{-\alpha} + e^0$$

$$= -\alpha e^{-\alpha} - e^{-\alpha} + 1$$

$$\lim_{\alpha \to \infty} \int_0^\alpha x e^{-x} dx = \lim_{\alpha \to \infty} \left[\underbrace{-\alpha e^{-\alpha}}_{?} - \underbrace{e^{-\alpha}}_{\downarrow 0} + 1\right]$$

__Note__ $\lim_{\alpha \to \infty} \alpha e^{-\alpha} = \lim_{\alpha \to \infty} \frac{\alpha}{e^\alpha}$

L'Hopital for $\frac{\infty}{\infty}$ type

$$= \lim_{\alpha \to \infty} \frac{\alpha'}{(e^\alpha)'}$$

$$= \lim_{\alpha \to \infty} \frac{1}{e^\alpha} = 0.$$

Therefore,

$$\bar{x} = \lim_{\alpha \to \infty} \int_0^\alpha x e^{-x} dx = 0 - 0 + 1 = 1$$

i.e. $\bar{x} = 1$

- Let's use $V = M_2 - \bar{x}^2$.

- Compute $M_2 = \int_0^\infty x^2 p(x) dx$.

$$= \int_0^\infty x^2 e^{-x} dx$$

$$= \lim_{\alpha \to \infty} \int_0^\alpha x^2 e^{-x} dx$$

$$\int_0^\alpha x^2 e^{-x} dx$$

Integration by parts

$u = x^2 \qquad dv = e^{-x} dx$

$$= x^2 \cdot (-e^{-x}) \Big]_0^\alpha - \int_0^\alpha 2x \cdot (-e^{-x}) dx$$

$$= -\alpha^2 e^{-\alpha} + 2 \underbrace{\int_0^\alpha x e^{-x} dx}$$

Integration by parts

$f = x \quad dg = e^{-x} dx$

$$= x(-e^{-x}) \Big]_0^\alpha - \int_0^\alpha 1 \cdot (-e^{-x}) dx$$

$$= -\alpha^2 e^{-\alpha} + 2 \left\{ -\alpha e^{-\alpha} + \left[-e^{-x} \right]_0^\alpha \right\}$$

$$= -\alpha^2 e^{-\alpha} + 2 \left\{ -\alpha e^{-\alpha} - e^{-\alpha} + 1 \right\}$$

$$= -\alpha^2 e^{-\alpha} - 2\alpha e^{-\alpha} - 2e^{-\alpha} + 2.$$

So, $M_2 = \lim_{\alpha \to \infty} \int_0^\alpha x^2 e^{-x} dx$

$$= \lim_{\alpha \to \infty} \left[\underbrace{-\alpha^2 e^{-\alpha}}_{\downarrow\, 0} \underbrace{- 2\alpha e^{-\alpha}}_{\downarrow\, 0} \underbrace{- 2e^{-\alpha}}_{\downarrow\, 0} + 2 \right]$$

$$= 2$$

Because.

- $\lim_{\alpha\to\infty} e^{-\alpha} = \lim_{\alpha\to\infty} \frac{1}{e^{\alpha}} = 0.$ because as $\alpha\to\infty$, $e^{\alpha}\to\infty$.

- $\lim_{\alpha\to\infty} \alpha e^{-\alpha} = \lim_{\alpha\to\infty} \frac{\alpha}{e^{\alpha}}$

$$= \lim_{\alpha\to\infty} \frac{(\alpha)'}{(e^{\alpha})'}$$ L'Hopital. for $\frac{\infty}{\infty}$ type

$$= \lim_{\alpha\to\infty} \frac{1}{e^{\alpha}} = 0$$

- $\lim_{\alpha\to\infty} \alpha^2 e^{-\alpha} = \lim_{\alpha\to\infty} \frac{\alpha^2}{e^{\alpha}}$

$$= \lim_{\alpha\to\infty} \frac{(\alpha^2)'}{(e^{\alpha})'}$$ L'Hopital for $\frac{\infty}{\infty}$ type

$$= \lim_{\alpha\to\infty} \frac{2\alpha}{e^{\alpha}}$$

$$= \lim_{\alpha\to\infty} \frac{(2\alpha)'}{(e^{\alpha})'}$$

$$= \lim_{\alpha\to\infty} \frac{2}{e^{\alpha}} = 0.$$

- Use $M_2 = 2$ & $\bar{x} = 1$

in the formula $V = M_2 - \bar{x}^2$

$\therefore V = 2 - 1^2 = 2 - 1$

$= 1.$

the variance $\underline{V = 1}$ ▨