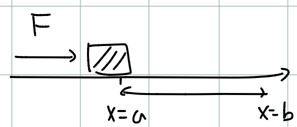
Lec 24 .

- Work §7.6.
- Probability §7.8.

Work . §7.6.

- single body



- When F is constant .

"work = force × distance"

$$W = F \cdot (b-a).$$

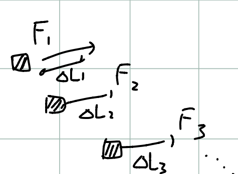
units
Force → N. Newton
distance → m meter.
work → N · m.

- when F is not constant . in general

$$dW = F(x)\, dx$$

$$W = \int_{x=a}^{x=b} dW = \int_a^b F(x)\, dx$$

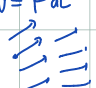
- Many bodies



$$W = \sum_{i=1}^{N} W_i$$
$$= \sum_{i=1}^{N} F_i \, \Delta L_i$$

- fluid.

$dW = F dL$

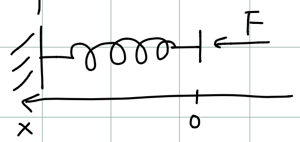


continuous family of "particles"

need
multi variable calculus.

(but, special cases can be handeled using single variable calculus.

Ex. spring.



$F(x) = kx$

Hooke's law. k = constant.

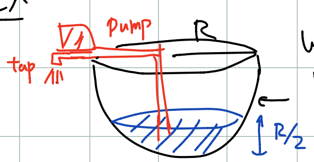
Suppose $k = 2000$

To stretch the spring by 3 cm.

$$W = \int_0^3 2000x\,dx = 1000x^2\Big]_0^3 = 9000\ N\cdot cm$$

$$= \underline{90\ N\cdot m}. \leftarrow 1\,cm = \tfrac{1}{100}\,m.$$

Ex



Water tank

hemi-sphere, radius R (m)

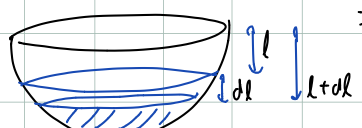
· gravitational constant g

· mass density per volume ρ ($\rho = 1000\,(kg/m^3)$)

To empty the tank, at least how much work has to be done by the pump?

<sol>. • Consider the infinitesimal moment

for the distance from the top of the container to the water surface, to be from l to $l+dl$.

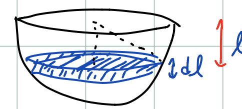


Then the amount of water corresponding the thin slice has to be moved $\underline{\text{at least}}$ distance l.

• Work corresponding to the slice at distance l from the top of the container.

$$dW = F \cdot l$$

• Force corresponding to the thin slice.

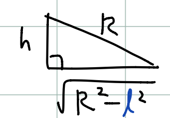


gravitational force = g · mass.

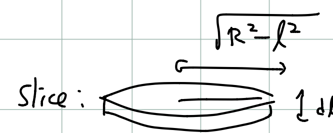
$$F = g \cdot \rho \cdot dV$$

gravitational constant.

mass density.



• volume $dV =$ area · width



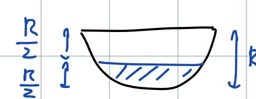
$$= \pi \left(\sqrt{R^2-l^2}\right)^2 dl.$$

$$= \pi (R^2-l^2)\, dl.$$

• So. $dW = g \cdot \rho \cdot \pi (R^2-l^2)\,dl \cdot l$

$$= g \cdot \rho \cdot \pi (R^2-l^2)\, l\, dl$$

• Total work



$$W = \int_{l=\frac{R}{2}}^{l=R} dW = \int_{l=\frac{R}{2}}^{l=R} g\rho\pi (R^2-l^2)\, l\, dl$$

$$= \underbrace{g\rho\pi}_{\text{constants.}} \int_{\frac{R}{2}}^{R} (R^2-l^2)\, l\, dl = g\rho\pi \left[R^2 \frac{l^2}{2} - \frac{l^4}{4} \right]_{\frac{R}{2}}^{R}$$

$$= g\rho\pi \left[\frac{R^4}{2} - \frac{R^4}{4} - \left(R^2 \cdot \frac{R^2}{8} - \frac{R^4}{2^4 \cdot 4} \right) \right]$$

$$= g\rho\pi R^4 \cdot \left(\frac{1}{4} - \frac{1}{8} - \frac{1}{64} \right) = g\rho\pi R^4 \cdot \frac{16-8-1}{64}$$

$$= \underline{\frac{7}{64} g\rho\pi R^4}.$$

Probability. §7.8. 3 lectures

· Exercises in §7.8 : 12, 15, 16, 18, 20, plus. Ch. review challenging problem #7.

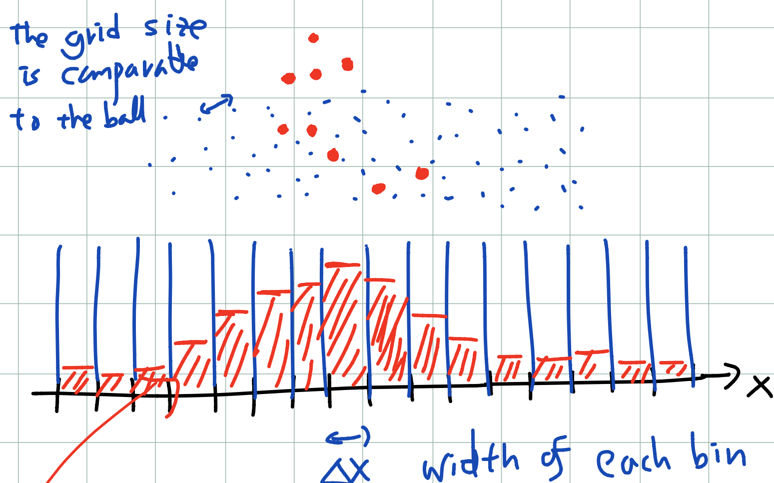
Continuous Probability Distributions

- From discrete probability to continuous probability.
- Probability density

§. From discrete probability to continuous probability.

Example for motivation

- Watch YouTube "Galton Board" (the one with 2:24 min).

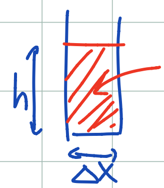


· Suppose each ball has small area A.

· Suppose the total area = 1.

(i.e. Total # of balls $= \frac{1}{A}$).

• In each bin,



• The area $= (\text{\# of balls in the bin}) \times \underbrace{A}$

$$= \frac{\text{\# of balls in the bin}}{\text{total \# of balls}}$$

$A = \dfrac{1}{\text{Total \# of balls}}$

So, this area represents
the probability for a ball to be in the bin.

• The height $h = \dfrac{\text{the area}}{\text{width}}$

$$= \boxed{\frac{\text{\# of balls in the bin}}{\text{total \# of balls}}} \Big/ \Delta X$$

The height represents how much portion of balls
per the unit width.

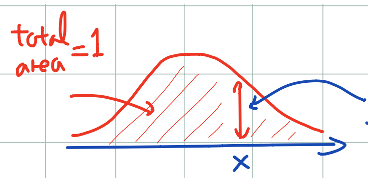
(i.e. the probability per unit width ΔX).

* This is a probability density!..

• Continuous Probability

If the size of each ball and the width of each bin
get smaller & smaller
(i.e. $A \to 0$ & $\Delta X \longrightarrow 0$).

Then, the resulting shape will be a curve.



this height represents the $\boxed{\text{probability density}}$ per unit length of a ball to arrive at x.

* Probability density is similar to mass density. (except the total probability is always 1.)

RMK In the Galton board, the curve is the bell curve. "Normal distribution" (Gaussian)

$$y = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2V}(x-\mu)^2}.$$

μ = constant. = mean of the distribution
V = constant. = variance of the "
σ = constant = standard deviation

We will learn these concepts in the next two lectures.

§ Probability density function:

Def. A function $p(x)$ defined on $[a, b]$ (here, a can be $-\infty$, b can be ∞) is said to be a probability density if

1. $p(x) \geq 0$ for all x
2. $\int_a^b p(x)\,dx = 1.$ ← $[a, b]$ is the possible values of x

* Probability density is similar to mass density. except that total probability = 1, ALWAYS!

Def $\boxed{\text{A variable } x \text{ with probability density } p(x) \text{ is called a \underline{random variable}}}$

e.g. $p(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ is a probability density function

because 1). $p(x) \geq 0$ for all x.

2). Fact $\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2} dx = 1$.

e.g. (Probability VS Probability density)

Let, in the Galton board example,

$p(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$

What is the probability for a small ball to arrive at $x = 1$?

<sol>. ~~At $x=1$, $p(1) = \frac{1}{\sqrt{\pi}} e^{-1}$~~

~~So, the probability is $\frac{1}{\sqrt{\pi}} e^{-1}$.~~

WRONG!

NOTE Probability density $\neq$ probability

(similarly to mass density $\neq$ mass).

At the point $x=1$, the width of the interval $= 0$.

So, the probability for a ball to arrive there is zero. ▨

More systematic way
to get probability from probability density:
Integrate the probability density!
(Just like how you get mass from mass density.)

- Probability from Probability density function.

a The probability for a random variable takes on values in the interval $\alpha \le x \le \beta$:

$$\underbrace{\mathbb{P}(\alpha \le X \le \beta)}_{\text{just a notation to denote the probability}} = \int_\alpha^\beta p(x)\,dx.$$

e.g. $p(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$

$$\mathbb{P}(x=1) = \int_1^1 p(x)\,dx = \int_1^1 \frac{1}{\sqrt{\pi}} e^{-x^2}\,dx = 0$$

b Ex. Let $p(x) = Cx^2$, $\underbrace{0 \le x \le 1}_{\text{possible values of } x.}$. C = const. be a probability density.

① Determine C.

② Find probability for x to be in $\frac{1}{2} \le x \le 1$.

<sol>

① $1 = \int_0^1 p(x)\,dx = C\int_0^1 x^2\,dx = C\left[\frac{x^3}{3}\right]_0^1 = C \cdot \frac{1}{3}$. $\therefore \underline{C = 3}$

② $\mathbb{P}(\frac{1}{2} \le x \le 1) = \int_{\frac{1}{2}}^1 p(x)\,dx = \int_{\frac{1}{2}}^1 \underbrace{3}_{C=3 \text{ from ①}} \cdot x^2\,dx = \left[x^3\right]_{\frac{1}{2}}^1 = 1 - \left(\frac{1}{2}\right)^3 = \underline{\frac{7}{8}}.$

Next lecture:
- Cumulative function
- mean (expectation).
- Some applications