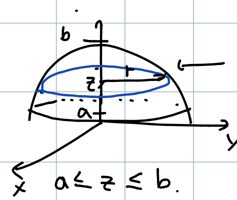
Lec 23

- Center of Mass §7.4 (surfaces of revolution)
- Work §7.6.

• Center of mass of 2-dim'l surfaces of revolution in 3D.



$a \le z \le b$.

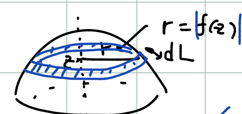
← surface S.

density per area

$\rho = \rho(z)$ (assume it depends only on z)

S = surface of revolution of a curve. $r = |f(z)|$ ← radius at z

Total mass: $M = \int_S dm$.



$$M = \int_{z=a}^{z=b} \rho(z)\, 2\pi |f(z)| \sqrt{1+[f'(z)]^2}\, dz.$$

mass density per area

the mass of the circular band: $dm = \rho(z)\cdot dS$

$$= \rho(z)\cdot 2\pi r \cdot dL$$

$$= \rho(z)\cdot 2\pi \underbrace{|f(z)|}_{r} \cdot \underbrace{\sqrt{1+[f'(z)]^2}\, dz}_{dL}$$

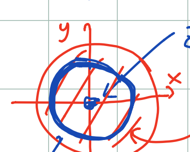
Center of Mass:

$M_{x_0=0} = 0$

$M_{y_0=0} = 0$

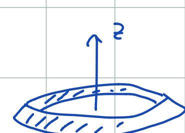
By symmetry: Because the density $\rho(z)$ & the shape of S are symmetric with respect to z-axis.

The view point from the top.



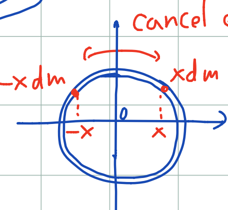
the surface

for each circular band



The contribution $x\,dm$ to the moment $M_{x_0=0}$ is cancelled by $-x\,dm$.

cancel each other.



because the shape and the density are symmetric about the y-axis.

Thus, $M_{x_0=0} = 0$., so $\bar{x} = 0$

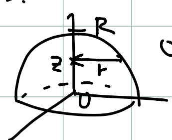
By the similar reason, $M_{y_0=0}$.. So, $\bar{y} = 0$.

$$M_{z_0=0} = \int_{z=a}^{z=b} z \cdot \rho(z)\, dS(z).$$

$$= \int_{z=a}^{z=b} z \cdot \rho(z) \cdot 2\pi \left|f(z)\right| \cdot \sqrt{1+\left[f'(z)\right]^2}\, dz$$

$$\bar{z} = \frac{M_{z_0=0}}{m}. \qquad \square$$

e.g.



$0 \le z \le R.$ $\quad r = \sqrt{R^2 - z^2} = f(z) \qquad f'(z) = \frac{-z}{\sqrt{R^2-z^2}}$

$\rho = \rho(z) = 1$ ← density per area.

$$m = \int_0^R 1 \cdot 2\pi \cdot \sqrt{R^2 - z^2} \sqrt{1 + \left[\frac{-z}{\sqrt{R^2-z^2}}\right]^2}\, dz$$

$$= \int_0^R 1 \cdot 2\pi \sqrt{R^2}\, dz = 2\pi R^2.$$

The shape and density are symmetric w.r.t. z-axis

So, $M_{x_0=0} = 0$, $M_{y_0=0} = 0$. $\Rightarrow \bar{x} = 0$, $\bar{y} = 0$.

$$M_{z_0=0} = \int_{z=0}^{z=R} z \cdot 1 \cdot 2\pi \cdot \sqrt{R^2}\, dz$$

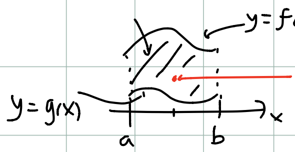
$$= \pi R \left[z^2\right]_0^R = \pi R^3.$$

$$\therefore \bar{z} = \frac{M_{z_0=0}}{m} = \frac{\pi R^3}{2\pi R^2} = \frac{R}{2}.$$

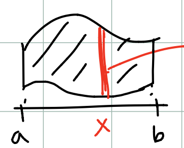
$$\therefore (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{R}{2}\right).$$

When $\rho \equiv 1$, the center of mass is called the centroid.

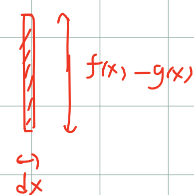
- Center of mass of a plate $a \le x \le b$, $g(x) \le y \le f(x)$, with density $\sigma(x)$ per area



density $\sigma(x)$ depends only on x.



same density $\sigma(x)$ along this vertical strip

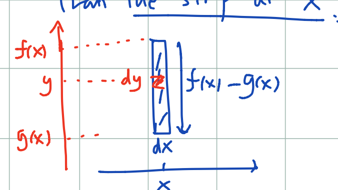


$$M_{x_0=0} = \int_{x=a}^{x=b} x \cdot \sigma(x) \cdot \underbrace{f(x) \cdot dx}_{\text{area of the vertical strip.}}$$

$$M_{y_0=0} = \text{"}\int y \cdot dm\text{"}$$

$$= \int_{x=a}^{x=b} \left[\int_{y=g(x)}^{y=f(x)} y\, \sigma(x)\, dy\, dx \right]$$

contribution to the y-moment $M_{y_0=0}$ from the strip at x:



$$= \int_{x=a}^{x=b} \left[\int_{y=g(x)}^{y=f(x)} y\, dy \right] \sigma(x)\, dx$$

$$= \int_{x=a}^{x=b} \left[\frac{y^2}{2} \right]_{y=g(x)}^{y=f(x)} \sigma(x)\, dx$$

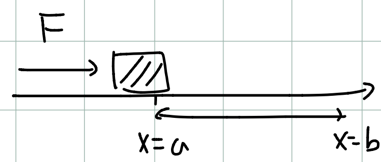
$$M_{x_0=0} = \int_{x=a}^{x=b} \frac{1}{2}\left([f(x)]^2 - [g(x)]^2 \right) \sigma(x)\, dx.$$

From these, can compute the center of mass $\bar{x} = \dfrac{M_{x_0=0}}{m}$, $\bar{y} = \dfrac{M_{y_0=0}}{m}$

$m = \text{total mass} = \int_a^b \sigma(x) f(x)\, dx.$ ∎

Work. §7.6.

- single body



- When F is constant.

"work = force × distance"

units
Force → N. Newton
distance → m meter.
work → N·m.

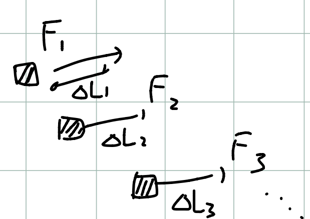
$$W = F \cdot (b-a).$$

- when F is not constant. in general

$$dW = F(x)\,dx$$

$$W = \int_{x=a}^{x=b} dW = \int_a^b F(x)\,dx$$

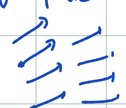
- Many bodies



$$W = \sum_{i=1}^{N} W_i = \sum_{i=1}^{N} F_i \,\Delta L_i$$

- fluid.

$dW = F\,dL$

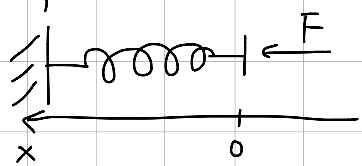


Continuous family of "particles"

need multi variable calculus.

(but, special cases can be handeled using single variable calculus.

Ex. spring.



$F(x) = kx$

Hooke's law. k = constant.

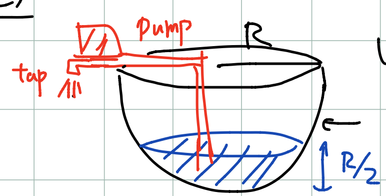
Suppose $k = 2000$

To stretch the spring by 3 cm.

$$W = \int_0^3 2000\,x\,dx = 1000x^2\Big]_0^3 = 9000 \text{ N}\cdot\text{cm}$$

$$= \underline{90 \text{ N}\cdot\text{m}}. \quad \leftarrow \quad 1\,\text{cm} = \tfrac{1}{100}\,\text{m}.$$

Ex



Water tank

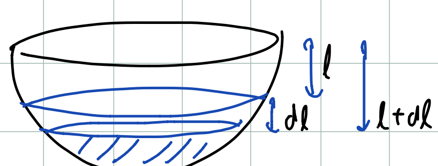
hemi-sphere, radius R (m)

. gravitational constant g

. mass density per volume ρ ($\rho = 1000\,(\text{kg}/\text{m}^3)$)

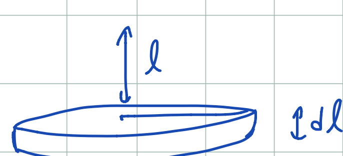
To empty the tank, at least how much work has to be done by the pump

<Sol>. • Consider the infinitesimal moment



for the distance from the top of the container to the water surface, to be from l to $l+dl$.

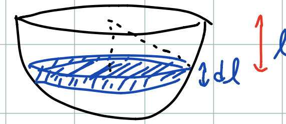
Then the amount of water corresponding the thin slice has to be moved at least distance l.



• Work corresponding to the slice at distance l from the top of the container.

$$dW = F \cdot l$$

- Force corresponding to the thin slice.

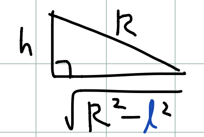


gravitational force $=$ g $\cdot$ mass.

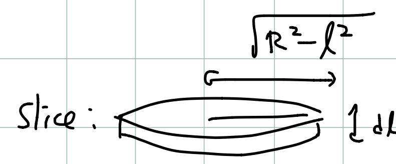
$$F = g \cdot \rho \cdot dV$$

g: gravitational constant.

ρ: mass density.



- volume $dV =$ area $\cdot$ width



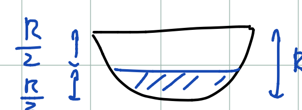
$$= \pi \left(\sqrt{R^2-l^2}\right)^2 dl.$$

$$= \pi (R^2-l^2)\, dl.$$

- So, $dW = g \cdot \rho \cdot \pi (R^2-l^2)\, dl \cdot l$

$$= g \cdot \rho \cdot \pi (R^2-l^2)\, l\, dl$$

- Total work



$$W = \int_{l=\frac{R}{2}}^{l=R} dW = \int_{l=\frac{R}{2}}^{l=R} g\rho\pi (R^2-l^2)\, l\, dl$$

$$= g\rho\pi \int_{\frac{R}{2}}^{R} (R^2-l^2)\, l\, dl = g\rho\pi \left[R^2 \frac{l^2}{2} - \frac{l^4}{4} \right]_{\frac{R}{2}}^{R}$$

constants.

$$= g\rho\pi \left[\frac{R^4}{2} - \frac{R^4}{4} - \left(R^2 \cdot \frac{R^2}{8} - \frac{R^4}{2^4 \cdot 4} \right) \right]$$

$$= g\rho\pi R^4 \cdot \left(\frac{1}{4} - \frac{1}{8} - \frac{1}{64} \right) = g\rho\pi R^4 \cdot \frac{16-8-1}{64}$$

$$= \frac{7}{64} g\rho\pi R^4.$$