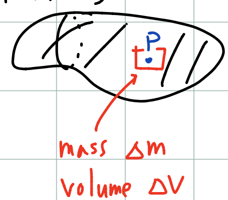
Lec 22 Mass, Moments, Centre of Mass § 7.4

(Read the textbook!)

- 3-D.

An inhomogeneous solid



Mass density per volume at P

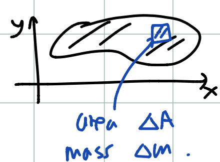
$$\rho(P) = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V}$$

$$\boxed{dm = \rho(P)\, dV.}$$

$$\text{mass} = \int dm = \int \rho(P)\, dV.$$

- 2D

surface



density per area

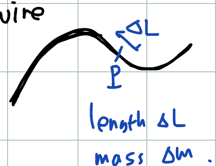
$$\sigma(P) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}$$

$$\boxed{dm = \sigma(P)\, dA.}$$

$$\text{mass} = \int dm = \int \sigma(P)\, dA$$

- 1 D

wire



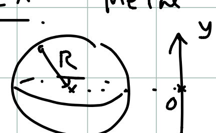
density per length

$$\delta(P) = \lim_{\Delta L \to 0} \frac{\Delta m}{\Delta L}$$

$$\boxed{dm = \delta(P)\, dL}$$

$$\text{mass} = \int dm = \int \delta(P)\, dL.$$

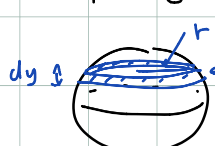
Ex. Metal ball.



density per volume $\rho(y) = |y|$

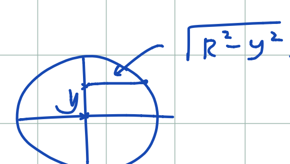
Mass = ?

<sol>. "$M = \int dm$". $dm = \rho(y)\, dV(y)$



$$dV = \pi [r(y)]^2 dy$$

$$= \pi (R^2 - y^2)\, dy.$$



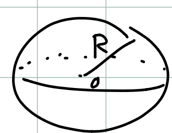
$$\therefore\ M = \int_{-R}^{R} \underbrace{|y|}_{\rho(y)} \cdot \underbrace{\pi(R^2 - y^2)\,dy}_{dV(y)}$$

$$= 2\int_0^R y \cdot \pi(R^2 - y^2)\,dy$$

$$= 2\pi\left[R^2\frac{y^2}{2} - \frac{y^4}{4}\right]_0^R$$

$$= 2\pi R^4 \cdot \frac{1}{4} = \frac{\pi R^4}{2}. \quad ▨$$

Ex.



r = distance from the origin.

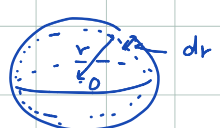
density per volume

$$\rho(r) = e^r$$

Mass = ?

(sol) $\quad dm = \rho(r)\,dV.$

Use the spherical shells, since ρ depends only on r, i.e. constant on each spherical shell.



$$dV = (\text{area of the sphere of radius } r) \cdot dr$$

$$= 4\pi r^2 \cdot dr.$$

$$\therefore\ \text{Mass} = \int_{r=0}^{r=R} dm = \int_0^R \underbrace{e^r}_{\rho(r)} \cdot \underbrace{4\pi r^2\,dr}_{dV}$$

$$= 4\pi\int_0^R r^2 e^r\,dr$$

$$= 4\pi\left\{\left[r^2 e^r\right]_0^R - \int_0^R 2r e^r\,dr\right\} \quad \leftarrow \text{Integration by parts.}$$

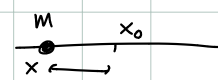
$$= 4\pi\left\{R^2 e^R - \left[2re^r\right]_0^R + \int_0^R 2e^r\,dr\right\}$$

$$= 4\pi\left\{R^2 e^R - 2Re^R + 2\cdot(e^R - 1)\right\} \quad ▨$$

• Moments & centre of Mass.

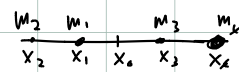
moment "=" weighted displacement from a given center

1D. Moment about the point $x = x_0$ of mass m



$= m \cdot$ (signed distance from x_0)

$= m\,(x - x_0)$

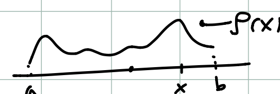


Total moments (about $x = x_0$) $= \sum_{k=1}^{N} m_k (x_k - x_0)$

Center of mass $\bar{x}$: the point where the moment about $x = \bar{x}$ is zero

i.e. $0 = \sum_{k=1}^{N} (x_k - \bar{x})\, m_k \quad \therefore \quad \bar{x} = \dfrac{\sum_{k=1}^{N} m_k x_k}{\sum_{k=1}^{N} m_k}$

Continuous mass distribution



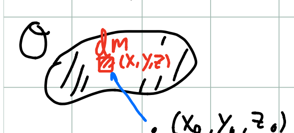
$\rho(x)$ mass density per length.

1D case:

Center of mass $= \dfrac{\int_a^b x\,\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$.

3D (similarly 2D)

An object (can be a curve/surface/solid)



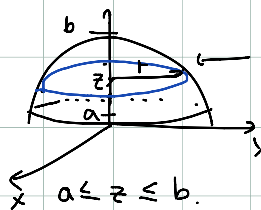
Total mass $M = \int_{\mathcal{O}} dm$. ← notation for the "sum" of each "small" mass dm over the object S.

The moment about (x_0, y_0, z_0) has three components.

$$\begin{cases} M_{x_0} = \int_{\mathcal{O}} (x - x_0)\, dm \\ M_{y_0} = \int_{\mathcal{O}} (y - y_0)\, dm \\ M_{z_0} = \int_{\mathcal{O}} (z - z_0)\, dm \end{cases}$$

Center of mass : $\bar{x} = \dfrac{M_{x_0=0}}{m}, \quad \bar{y} = \dfrac{M_{y_0=0}}{m}, \quad \bar{z} = \dfrac{M_{z_0=0}}{m}$. ← Here $(x_0, y_0, z_0) = (0,0,0)$

● Center of mass of 2-dim'l surfaces of revolution in 3D.

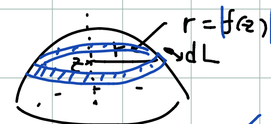


surface S.
density per area
$\rho = \rho(z)$ (assume it depends only on z)

$a \le z \le b$.

S = surface of revolution of a curve. $r = |f(z)|$ ← radius at z

Total mass: $M = \int_S dm$.



$$M = \int_{z=a}^{z=b} \rho(z)\, 2\pi |f(z)| \sqrt{1+[f'(z)]^2}\, dz.$$

mass density per area

$dm = \rho(z) \cdot dS$ ← the mass of the circular band

$= \rho(z) \cdot 2\pi r \cdot dL$

$= \rho(z) \cdot 2\pi \underbrace{|f(z)|}_{r} \cdot \underbrace{\sqrt{1+[f'(z)]^2}\, dz}_{dL}$

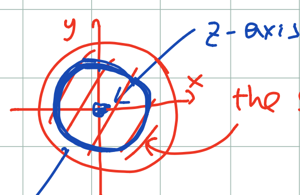
Center of Mass:

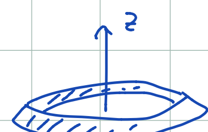
$M_{x_0=0} = 0$

$M_{y_0=0} = 0$

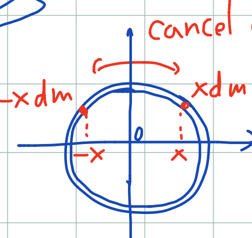
By symmetry: Because the density $\rho(z)$ & the shape of S are symmetric with respect to z-axis.

The view point from the top.





for each circular band
The contribution
$x\,dm$ to the moment $M_{x_0=0}$
is cancelled by $-x\,dm$.



cancel each other, because the shape and the density are symmetric about the y-axis.

Thus, $M_{x_0=0} = 0$, so $\bar{x} = 0$

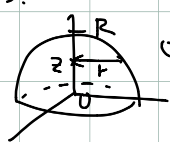
By the similar reason, $M_{y_0=0}$. So, $\bar{y} = 0$.

$$M_{z_0=0} = \int_{z=a}^{z=b} z \cdot \rho(z)\, dS(z).$$

$$= \int_{z=a}^{z=b} z \cdot \rho(z) \cdot 2\pi |f(z)| \cdot \sqrt{1 + [f'(z)]^2}\, dz$$

$$\bar{z} = \frac{M_{z_0=0}}{m}. \quad \boxed{//}$$

e.g.



$0 \le z \le R$. $r = \sqrt{R^2 - z^2} = f(z)$ $f'(z) = \frac{-z}{\sqrt{R^2-z^2}}$

$\rho = \rho(z) = 1$ ← density per area.

$$m = \int_0^R 1 \cdot 2\pi \cdot \sqrt{R^2-z^2} \sqrt{1 + \left[\frac{-z}{\sqrt{R^2-z^2}}\right]^2}\, dz$$

$$= \int_0^R 1 \cdot 2\pi \sqrt{R^2}\, dz = 2\pi R^2.$$

The shape and density are symmetric w.r.t. z-axis

So, $M_{x_0=0} = 0$, $M_{y_0=0} = 0$. $\Rightarrow$ $\bar{x} = 0$, $\bar{y} = 0$.

$$M_{z_0=0} = \int_{z=0}^{z=R} z \cdot 1 \cdot 2\pi \cdot \sqrt{R^2}\, dz$$

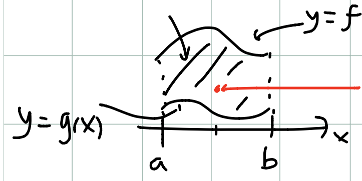
$$= \pi R \left[z^2\right]_0^R = \pi R^3.$$

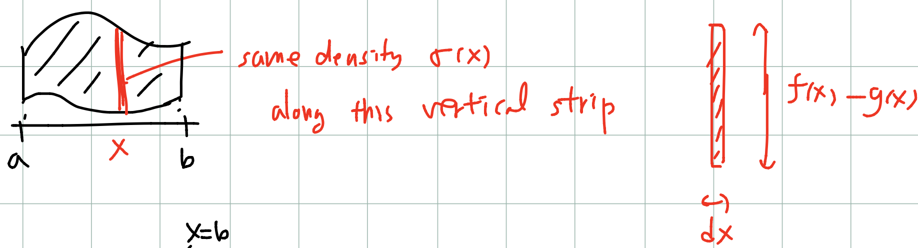
$$\therefore \bar{z} = \frac{M_{z_0=0}}{m} = \frac{\pi R^3}{2\pi R^2} = \frac{R}{2}.$$

$$\therefore (\bar{x}, \bar{y}, \bar{z}) = (0, 0, \tfrac{R}{2}).$$

When $\rho \equiv 1$, the center of mass is called the <u>centroid</u>.

- Center of mass of a plate $a \le x \le b$, $g(x) \le y \le f(x)$ with density $\sigma(x)$ per area





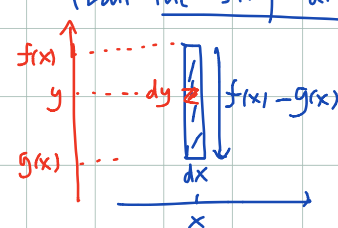
$$M_{x_0=0} = \int_{x=a}^{x=b} x \cdot \sigma(x) \cdot \underbrace{f(x) \cdot dx}_{\text{area of the vertical strip.}}$$

$$M_{y_0=0} = \text{"} \int y \cdot dm \text{"}$$

$$= \int_{x=a}^{x=b} \left[\int_{y=g(x)}^{y=f(x)} y\, \sigma(x)\, dy\, dx \right]$$

contribution to the y-moment $M_{y_0=0}$ from the strip at x.

$$= \int_{x=a}^{x=b} \left[\int_{y=g(x)}^{y=f(x)} y\, dy \right] \sigma(x)\, dx$$



$$= \int_{x=a}^{x=b} \left[\frac{y^2}{2} \right]_{y=g(x)}^{y=f(x)} \sigma(x)\, dx$$

$$M_{x_0=0} = \int_{x=a}^{x=b} \frac{1}{2}\left([f(x)]^2 - [g(x)]^2 \right) \sigma(x)\, dx.$$

From these, can compute the center of mass $\bar{x} = \frac{M_{x_0=0}}{m}$, $\bar{y} = \frac{M_{y_0=0}}{m}$

m = total mass = $\int_a^b \sigma(x) f(x)\, dx$. ▨