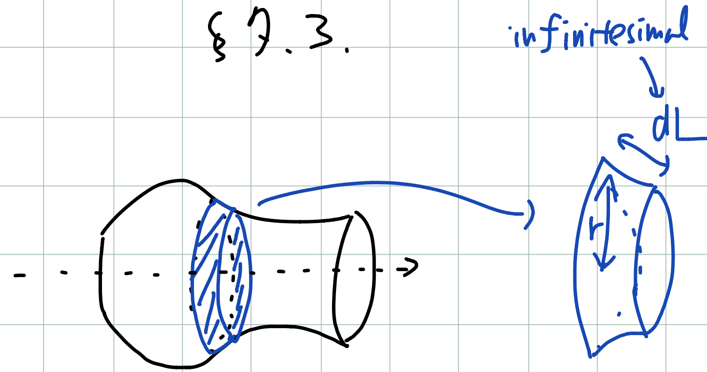
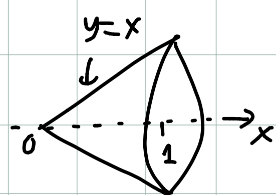
Lec 21. Surface areas of surfaces of revolution.

§ 7.3.

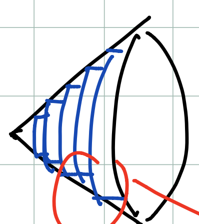


infinitesimal surface area

$$dS = 2\pi r \cdot dL$$

Ex 

$$S = \int_{x=0}^{x=1} dS = \int_{x=0}^{x=1} 2\pi x \sqrt{1+1^2}\, dx = 2\pi \frac{x^2}{2} \sqrt{2} \Big]_0^1 = \sqrt{2}\pi$$

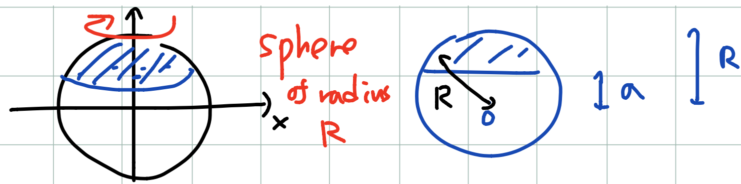


Note: the "staircase shape" approximation does NOT give the surface area.

this error for the surface area cannot be eliminated even after infinitely many subdivisions.

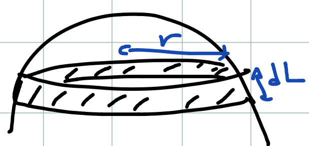
※ For the volume, this "staircase shape" approximation is what we use, and it does work since the error for volume will eventually be eliminated after taking infinitely many small subdivisions.

Ex.



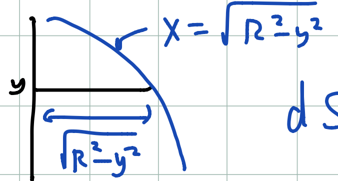
Surface area of the spherical cap?

<sol>.



$$dS = 2\pi r \, dL$$

Method ① Use y variable.



$$\frac{dx}{dy} = \frac{y}{\sqrt{R^2-y^2}}$$

$$dS = 2\pi\sqrt{R^2-y^2}\, dL(y)$$

$$= 2\pi\sqrt{R^2-y^2}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$dL = \sqrt{dx^2+dy^2}$

$= \sqrt{\left(\frac{dx}{dy}\right)^2+1}\, dy$

$$= 2\pi\sqrt{R^2-y^2}\sqrt{1+\frac{y^2}{R^2-y^2}}\, dy$$

$$= 2\pi\sqrt{R^2-y^2+y^2}\, dy$$

Surface area

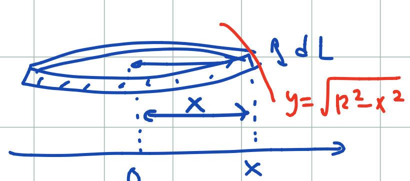
$$= 2\pi\sqrt{R^2}\,dy = 2\pi R\, dy$$

$$S = \int_{y=a}^{y=R} dS = \int_a^R 2\pi R\, dy = 2\pi \cdot R \; y\Big]_a^R = 2\pi R(R-a)$$

Method ② Use x-variable.

$dL = \sqrt{dx^2+dy^2}$

$= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$.



$$dS = 2\pi x \, dL$$

$$dS = 2\pi x \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{R^2-x^2}}$$

$$= 2\pi x\sqrt{1+\frac{x^2}{R^2-x^2}}\, dx$$

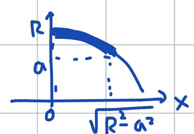
$$\therefore\ dS = \frac{2\pi x}{\sqrt{R^2-x^2}}\sqrt{R^2-x^2+x^2}\,dx$$

$$= \frac{2\pi x}{\sqrt{R^2-x^2}}\cdot R\,dx$$

Surface area

$$S = \int_{x=0}^{x=\sqrt{R^2-a^2}} dS = \int_{x=0}^{x=\sqrt{R^2-a^2}} 2\pi R\,\frac{x}{\sqrt{R^2-x^2}}\,dx$$

$x=\sqrt{R^2-y^2}$



$$= \int_{R^2}^{a^2} -\pi R\,\frac{du}{\sqrt{u}}$$

$u = R^2-x^2$

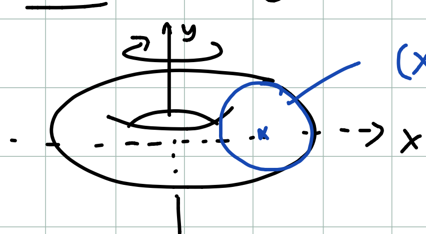
$du = -2x\,dx$

$$= -2\pi R\sqrt{u}\,\Big]_{R^2}^{a^2}$$

$$= 2\pi R\,[R-a]$$

∎

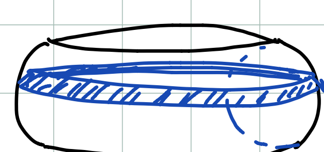
Ex doughnut.



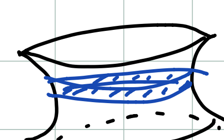
$(x-b)^2+y^2=a^2$

Surface area = ?

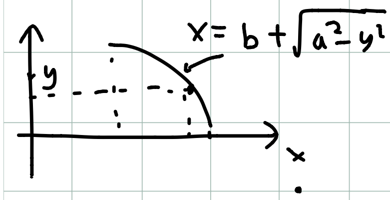
<sol>. Two parts: outer part



inner part:



Outer part : $dS_o = 2\pi r \, dL$

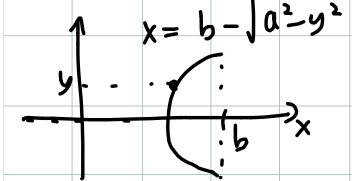


$$= 2\pi x \cdot \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$= 2\pi \left(b + \sqrt{a^2 - y^2}\right) \sqrt{1 + \frac{y^2}{a^2 - y^2}} \, dy$$

$$= 2\pi \left(\frac{b}{\sqrt{a^2 - y^2}} + 1\right) a \cdot dy.$$

Inner part.



$$dS_I = 2\pi r \, dL$$

$$= 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$= 2\pi \left(b - \sqrt{a^2 - y^2}\right) \sqrt{1 + \frac{y^2}{a^2 - y^2}} \, dy$$

$$= 2\pi \left(\frac{b}{\sqrt{a^2 - y^2}} - 1\right) \cdot a \, dy$$

Total surface area:

$$S = S_o + S_I = \int_{y=-a}^{y=a} (dS_o + dS_I)$$

$$= \int_{y=-a}^{y=a} \left[2\pi a \cdot \left(\frac{b}{\sqrt{a^2 - y^2}} \cancel{+1}\right) dy + 2\pi a \cdot \left(\frac{b}{\sqrt{a^2 - y^2}} \cancel{-1}\right) dy\right]$$

$$= 4\pi a b \int_{y=-a}^{y=a} \frac{1}{\sqrt{a^2 - y^2}} \, dy$$

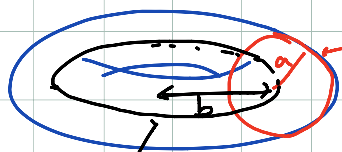
$$= 4\pi a b \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = 4\pi^2 ab.$$

$$\int_{-a}^{a} \frac{1}{\sqrt{a^2 - y^2}} \, dy \qquad y = a \sin\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a \cos\theta}{a |\cos\theta|} \, d\theta \qquad dy = a \cos\theta \, d\theta$$

$$\sqrt{a^2 - y^2} = \sqrt{a^2(1 - \sin^2\theta)} = a\sqrt{\cos^2\theta} = a|\cos\theta|.$$

Q. Can you see why the area is $(2\pi a)\cdot(2\pi b)$? Think about this.



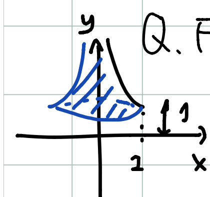
length of this circle $= 2\pi a$

length of this circle $= 2\pi b$.

Ex (surface area of an unbounded surface.)

$p>0$, $y = \dfrac{1}{x^p}$ $\quad 0 < x \le 1$. revolved about the y-axis.

Q. For what $p>0$, is the surface area finite?



Surface area

$$= \int_{y=1}^{y=\infty} 2\pi x \cdot dL$$

$$x = \frac{1}{y^{\frac{1}{p}}}$$

$$= \int_{y=1}^{y=\infty} 2\pi \cdot \frac{1}{y^{1/p}} \cdot dL$$

$$= \int_{y=1}^{y=\infty} 2\pi \cdot \frac{1}{y^{\frac{1}{p}+\frac{1}{p}+1}} \sqrt{\frac{1}{p^2} + y^{2(\frac{1}{p}+1)}}\, dy$$

$$= 2\pi \int_1^\infty \frac{1}{y^{\frac{2}{p}+1}} \sqrt{\frac{1}{p^2} + y^{2(\frac{1}{p}+1)}}\, dy \overset{\text{let}}{=} I$$

$$\boxed{\begin{aligned} dL &= \sqrt{dx^2+dy^2} \\ &= \sqrt{\left(\frac{dx}{dy}\right)^2+1}\, dy \\ &= \sqrt{\left(\frac{-\frac{1}{p}}{y^{\frac{1}{p}+1}}\right)^2+1}\, dy \\ &= \sqrt{\frac{1}{p^2}\frac{1}{y^{2(\frac{1}{p}+1)}}+1}\, dy \\ &= \frac{1}{y^{(\frac{1}{p}+1)}}\sqrt{\frac{1}{p^2}+y^{2(\frac{1}{p}+1)}}\, dy \end{aligned}}$$

Now, note for $p \geq 0$,
$$\frac{1}{y^{\frac{2}{p}+1}}\sqrt{\frac{1}{p^2} + y^{2(\frac{1}{p}+1)}} \geq \frac{1}{y^{\frac{2}{p}+1}}\sqrt{y^{2(\frac{1}{p}+1)}} = \frac{1}{y^{\frac{1}{p}}}.$$

for large enough y,

so that $y^{2(\frac{1}{p}+1)} \geq \frac{1}{p^2}$,
$$\frac{1}{y^{\frac{2}{p}+1}}\sqrt{\frac{1}{p^2}+y^{2(\frac{1}{p}+1)}} \le \frac{\sqrt{2\, y^{2(\frac{1}{p}+1)}}}{y^{2/p+1}} = \frac{\sqrt{2}}{y^{\frac{1}{p}}}.$$

From this & the comparison theorem for improper integrals,

the improper integral I converges

exactly when $\int_1^\infty \frac{1}{y^{1/p}}\,dy$ converges.

thus, exactly when $\frac{1}{p} > 1$. i.e. $0 < p < 1$.

$\blacksquare$