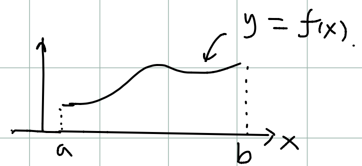
Lec 20. § 7.3. Arc length & Surface area.

Arc length.



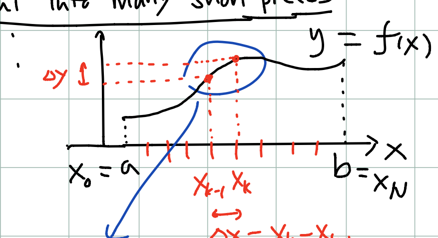
How to compute the arc-length?

Idea: ① divide into short pieces

② compute/approximate the length of short pieces.

③ Add up the lengths of short pieces.

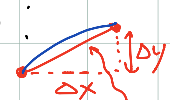
① Cut into many short pieces



$\Delta y = f(x_k) - f(x_{k-1})$

$\Delta x = x_k - x_{k-1}$

② :



line segment.

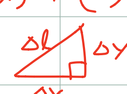
$$\boxed{\text{length of short arc } \Delta L \approx \text{length of the line segment } \Delta \ell}$$

length of the line segment

$$\Delta \ell = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

by Pythagoras law.

$$\left((\Delta \ell)^2 = (\Delta x)^2 + (\Delta y)^2 \right)$$



$\therefore$ Length of the short arc

$$\boxed{\Delta L \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

③ Add-up

Total length

length of k-th short arc.

$$L = \sum_{k=1}^{N} \Delta L \approx \sum_{k=1}^{N} \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Here, strictly speaking, we should denote ΔL_k, Δx_k, Δy_k to indicate that they correspond to k-th short arc. We omit k for simplicity in writing.

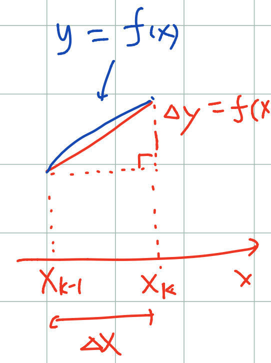
So, $$L = \boxed{\lim_{N \to \infty} \sum_{k=1}^{N} \sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$\Delta y = f(x_k) - f(x_{k-1})$ for the k-th short arc.

How to compute this limit?

Idea: Express the sum as a Riemann sum. then express the limit as an integral!

- $$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}} \cdot \Delta x$$



Observe: $$\boxed{\frac{\Delta y}{\Delta x} \approx f'(x_k)}$$

(because $f(x_k) - f(x_{k-1}) \approx f'(x_k)(x_k - x_{k-1})$)

$\therefore$ $$\boxed{\sqrt{(\Delta x)^2+(\Delta y)^2} \approx \sqrt{1+[f'(x_k)]^2}\cdot \Delta x}$$

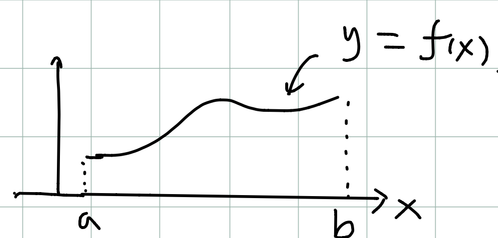
Thus, $$\sum_{k=1}^{N}\sqrt{(\Delta x)^2+(\Delta y)^2} \approx \sum_{k=1}^{N}\sqrt{1+[f'(x_k)]^2}\,\Delta x.$$

So,

total length $$L = \lim_{N\to\infty}\sum_{k=1}^{N}\sqrt{1+[f'(x_k)]^2}\,\Delta x$$

$$= \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

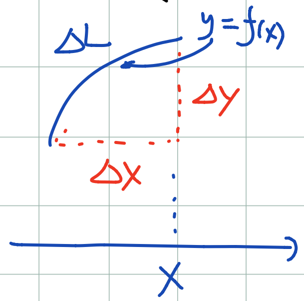
<u>Summary</u>



arc-length of the curve $y=f(x)$, $a\le x\le b$:

$$\boxed{L = \int_a^b \sqrt{1+[\underbrace{f'(x)}_{\text{derivative}}]^2}\,dx}$$

Remember!



length of the short arc.

$$\boxed{\Delta L \approx \sqrt{1+[f'(x)]^2}\ \Delta x}$$

"Infinitesimally"

$$dL = \sqrt{1+[f'(x)]^2}\ dx$$

This is a formal expression to indicate that for a very short interval of x with infinitely small length dx the arc length dL, which is also infinitely small is given by $\sqrt{1+[f'(x)]^2} \cdot dx$.

$$L = \underbrace{\int_a^b dL}_{} = \int_a^b \sqrt{1+[f'(x)]^2}\ dx.$$

The integral is the sum of infinitesimal arc-lengths.

• EX. Let $L(t)$ be the arc-length of the curve $y=e^x$, for $0 \le x \le t$.

Find $\dfrac{dL}{dt}$.

<sol>. arc-length.

$$L(t) = \int_0^t \sqrt{1+[y']^2}\ dx$$

$$= \int_0^t \sqrt{1+(e^x)^2}\ dx.$$

$$\frac{dL}{dt} = \sqrt{1+(e^t)^2} \quad \text{by the F.T.C.}$$

$$= \sqrt{1+e^{2t}}$$ ☒

Computing $I = \int \sqrt{1+e^{2x}}\,dx$

Try $u = 1+e^{2x}$. $du = 2e^{2x}dx = 2(u-1)\,dx$. $\therefore dx = \frac{1}{2(u-1)}du$

$\therefore$ $I = \int \sqrt{u} \cdot \frac{du}{2(u-1)}$. $v = \sqrt{u}$ $dv = \frac{1}{2\sqrt{u}}du = \frac{1}{2v}du$

$\therefore du = 2v\,dv$

$$= \int v \cdot \frac{2v\,dv}{2(v^2-1)} = \int \frac{v^2\,dv}{v^2-1} = \int \frac{v^2-1+1}{v^2-1}dv$$

$$= \int \left[1 + \frac{1}{v^2-1}\right]dv = v + \frac{1}{2}\int\left(\frac{1}{v-1} - \frac{1}{v+1}\right)dv$$

$$= v + \frac{1}{2}\ln|v-1| - \frac{1}{2}\ln|v+1| + C.$$ $v = \sqrt{u} = \sqrt{e^{2x}+1}$

$$= \sqrt{e^{2x}+1} + \frac{1}{2}\ln\left|\sqrt{e^{2x}+1}-1\right| - \frac{1}{2}\ln\left|\sqrt{e^{2x}+1}+1\right| + C.$$

Let us
Check: differentiate the last expression.

get $\frac{\cancel{2}e^{2x}}{\cancel{2}\sqrt{e^{2x}+1}} + \frac{1}{2}\cdot\frac{\frac{1}{\cancel{2}}\frac{\cancel{2}e^{2x}}{\sqrt{e^{2x}+1}}}{\sqrt{e^{2x}+1}-1} - \frac{1}{2}\cdot\frac{\frac{1}{\cancel{2}}\frac{\cancel{2}e^{2x}}{\sqrt{e^{2x}+1}}}{\sqrt{e^{2x}+1}+1}$.

$$= \frac{e^{2x}}{\sqrt{e^{2x}+1}}\cdot\left[\frac{(\sqrt{e^{2x}+1}-1)(\sqrt{e^{2x}+1}+1) + \frac{1}{2}(\sqrt{e^{2x}+1}+1) - \frac{1}{2}(\sqrt{e^{2x}+1}-1)}{(\sqrt{e^{2x}+1}-1)(\sqrt{e^{2x}+1}+1)}\right]$$

$$= \frac{\cancel{e^{2x}}}{\sqrt{e^{2x}+1}}\;\frac{e^{2x}\cancel{+1}\cancel{-1}+1}{\cancel{e^{2x}}\cancel{+1}\cancel{-1}} = \sqrt{e^{2x}+1}\ \checkmark$$

Ex arc-length for $y = x^2$. $0 \le x \le 1$

$\therefore$ $y' = 2x$.

$$\int_0^1 \sqrt{1+(2x)^2}\,dx = \int_0^2 \sqrt{1+u^2}\,\frac{du}{2}$$ $u = 2x$, $du = 2dx$.

note $\sec\theta > 0$ for $0 < \theta < \tan^{-1}2$

$$= \frac{1}{2}\int_0^{\tan^{-1}2} \sec\theta \cdot \sec^2\theta\,d\theta$$

$u = \tan\theta$. $du = \sec^2\theta\,d\theta$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$1+u^2 = \sec^2\theta$

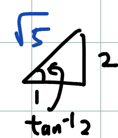
$$= \int_0^{\tan^{-1}2} \frac{1}{\cos^3\theta}\, d\theta$$

$$= \int_0^{\tan^{-1}2} \frac{\cos\theta}{\cos^4\theta}\, d\theta = \int_0^{\tan^{-1}2} \frac{\cos\theta\, d\theta}{(1-\sin^2\theta)^2}$$

$$= \int_0^{\frac{2}{\sqrt5}} \frac{dv}{(1-v^2)^2}$$

$v = \sin\theta$

$dv = \cos\theta\, d\theta$



$\sin(\tan^{-1}2) = \frac{2}{\sqrt5}$

$$= \int_0^{\frac{2}{\sqrt5}} \frac{dv}{(1-v)^2(1+v)^2} = \int_0^{\frac{2}{\sqrt5}} \left[\frac{A}{(1-v)} + \frac{B}{(1-v)^2} + \frac{C}{(1+v)} + \frac{D}{(1+v)^2}\right] dv$$

$B = \frac{1}{(1+1)^2}.$

$D = \frac{1}{(1-(-1))^2}$

$A(1-v)' = \left(\frac{1}{(1+v)^2}\right)'\Big|_{v=1} = -2(1+v)^{-3}\Big|_{v=1} = -\frac{1}{4}.$

$-A = -\frac14. \quad \therefore A = \frac14.$

$C(1+v)' = \left(\frac{1}{(1-v)^2}\right)'\Big|_{v=-1} = +2(1-v)^{-3}\Big|_{v=-1} = \frac14.$

$\therefore C = \frac14.$

$\therefore$ The integral is

$$\left[-\frac14 \ln|1-v| + \frac14 (1-v)^{-1} + \frac14 \ln|1+v| - \frac14 (1+v)^{-1}\right]_0^{\frac{2}{\sqrt5}}$$

$$= \frac14\left[\ln\left|\frac{1+v}{1-v}\right| + \frac{2v}{(1-v^2)}\right]_0^{\frac{2}{\sqrt5}} = \frac14\left(\ln\left|\frac{1+\frac{2}{\sqrt5}}{1-\frac{2}{\sqrt5}}\right| + \frac{2\cdot\frac{2}{\sqrt5}}{(1-\frac45)} - 0 - 0\right)$$

$$= \frac14\left(\ln\left(\frac{\sqrt5+2}{\sqrt5-2}\right) + 4\sqrt5\right) = \frac14 \ln\left|\frac{\sqrt5+2}{\sqrt5-2}\right| + \sqrt5.$$

$$\underline{\text{Another method}} \quad I = \int_0^{\tan^{-1}2} \sec^3\theta\, d\theta = \int_0^{\tan^{-1}2} \underbrace{\sec\theta}_{u} \cdot \underbrace{\sec^2\theta}_{v'} \, d\theta \qquad (\tan\theta)' = \sec^2\theta$$

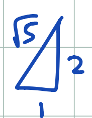
Integration by part

$$= \underbrace{\sec\theta}_{u}\ \underbrace{\tan\theta}_{v}\Big]_0^{\tan^{-1}2} - \int_0^{\tan^{-1}2} \underbrace{\sec\theta\tan\theta}_{u'}\ \underbrace{\tan\theta}_{v}\, d\theta \qquad (\sec\theta)' = \sec\theta\tan\theta$$

$$= \sec\theta\tan\theta\Big]_0^{\tan^{-1}2} - \int_0^{\tan^{-1}2} \sec\theta\,(\sec^2\theta - 1)\, d\theta$$

$$= \sec\theta\tan\theta\Big]_0^{\tan^{-1}2} - I + \int_0^{\tan^{-1}2} \sec\theta\, d\theta$$

$$\therefore\ 2I = \sec\theta\tan\theta\Big]_0^{\tan^{-1}2} + \int_0^{\tan^{-1}2} \sec\theta\, d\theta$$



$$= \sqrt{5}\cdot 2 + \left[\frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|\right]_0^{\tan^{-1}2}$$

$$= 2\sqrt{5} + \frac{1}{2}\cdot\ln\left|\frac{1+\frac{2}{\sqrt{5}}}{1-\frac{2}{\sqrt{5}}}\right|$$

$$= 2\sqrt{5} + \frac{1}{2}\ln\left(\frac{\sqrt{5}+2}{\sqrt{5}-2}\right)$$

$$\therefore\ I = \sqrt{5} + \frac{1}{4}\ln\left|\frac{\sqrt{5}+2}{\sqrt{5}-2}\right|.$$

$$\int \sec\theta\, d\theta = \int \frac{\cos\theta}{\cos^2\theta}\, d\theta = \int \frac{\cos\theta}{1-\sin^2\theta}\, d\theta = \int \frac{du}{1-u^2} \qquad u = \sin\theta$$

$$= \frac{1}{2}\int\left[\frac{1}{1-u} + \frac{1}{1+u}\right] du = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C = \frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + C$$

$\blacksquare$

This square root make computation difficult.

Often, $\int \sqrt{1+[f'(x)]^2}\,dx$ is very difficult to evaluate.

(Sometimes, only possible with numerical methods)

<u>Ex</u>. Arc-length of $y = \sin(x)$, $0 \le x \le \pi/2$.

$$L = \int_0^{\pi/2} \sqrt{1 + [y']^2}\,dx \qquad y' = \cos(x).$$

$$= \int_0^{\pi/2} \sqrt{1+\cos^2(x)}\,dx$$ ← need a computer....