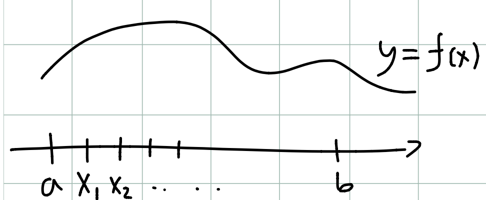
Lec 17 Numerical Integration §6.6, 6.7

Today: Introduction to $\left\{\begin{array}{l}\text{mid-point rule}\\ \text{trapezoid rule}\\ \text{Simpson's rule}\end{array}\right.$

Tue: Midterm 1.
Wed: error estimates §6.6.
Friday: §7.1. Volume.

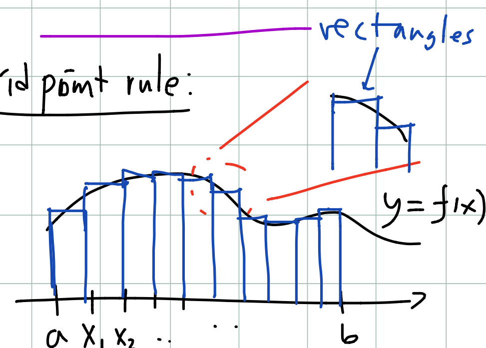
Motivation.
For many practical situations
- the functions are very complicated
- or they are given with no explicit expression (e.g. data from experiments)
- Integrals are important, but often approximations are sufficient.
- One may want to design an algorithm to compute relevant quantities, including integrals.
- Better algorithms are the faster, the more accurate

• We will discuss a few very basic algorithms & will consider error estimates.

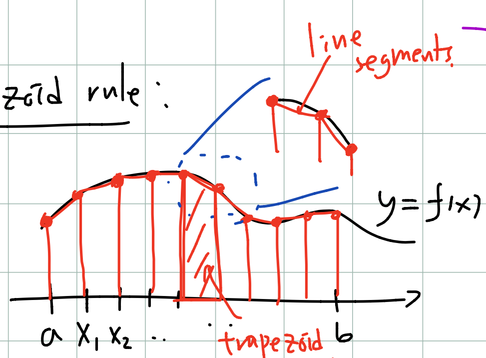


$$\Delta x = h = \frac{b-a}{N}$$

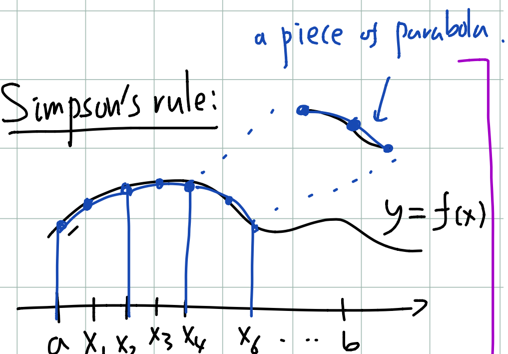
mid point rule:



trapezoid rule:



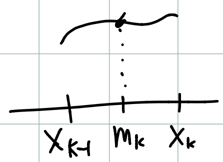
Simpson's rule:



- **Mid point rule.**

is a special type of Riemann sum

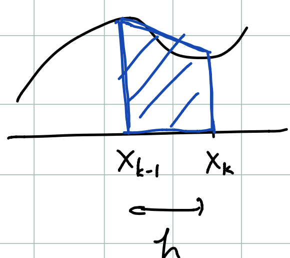
$$\int_a^b f(x)\,dx \approx M_N := \sum_{k=1}^{N} f\left(\frac{x_{k-1}+x_k}{2}\right)\cdot h$$

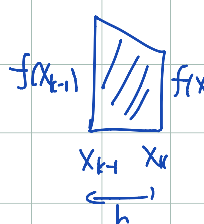


$m_k = \dfrac{x_{k-1}+x_k}{2}$

$$= h\cdot\left(f(m_1)+\cdots+f(m_N)\right)$$

- **trapezoid rule:**





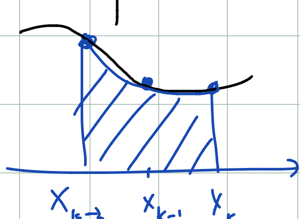
"area" $= \dfrac{f(x_{k-1})+f(x_k)}{2}\cdot h$

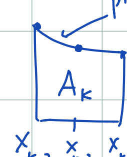
$$\int_a^b f(x)\,dx \approx T_N = \sum_{k=1}^{N}\left[\frac{f(x_{k-1})+f(x_k)}{2}\right]\cdot h$$

$$T_N = h\left(\frac{f(x_0)+f(x_1)}{2}+\frac{f(x_1)+f(x_2)}{2}+\cdots+\frac{f(x_{N-1})+f(x_N)}{2}\right)$$

$$= h\cdot\left[\frac{f(x_0)}{2}+f(x_1)+\cdots+f(x_{N-1})+\frac{f(x_N)}{2}\right]$$

- **Simpson's rule**

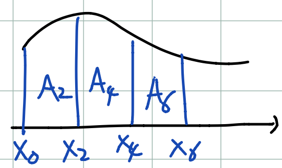




piece of parabola passing through $(x_{k-2}, f(x_{k-2}))$ $(x_{k-1}, f(x_{k-1}))$ $(x_k, f(x_k))$

$$\int_a^b f(x)\,dx \approx S_N = \text{sum of } A_k, \quad k = 2, 4, 6, \cdots, N$$

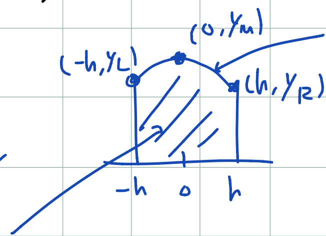
let's assume N = even.



Formula?

parabola $y = A + Bx + Cx^2$.

For each A_k,



Want to get formula for this "area" using y_L, y_M, y_R & h.

$$\int_{-h}^{h} (A + Bx + Cx^2)\,dx = \left[Ax + \frac{Bx^2}{2} + \frac{Cx^3}{3} \right]_{-h}^{h}$$

$$= 2hA + \frac{2C}{3} h^3$$

Note

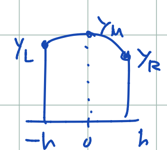
$y_L = A - Bh + Ch^2$ $\quad$; $\quad$ So, $\quad y_L + y_R = 2\underbrace{A}_{y_M} + 2Ch^2$

$y_M = A$

$y_R = A + Bh + Ch^2$

$$\therefore \frac{2Ch^3}{3} = \frac{y_L + y_R - 2y_M}{3} \cdot h$$

$\therefore$ the area



$$= h \left[2y_M + \frac{y_L + y_R - 2y_M}{3} \right]$$

$$= h \left[\frac{y_L + 4y_M + y_R}{3} \right]$$

So, letting $y_k = f(x_k)$

$$S_N = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) + \frac{h}{3}\left(y_2 + 4y_3 + y_4\right) + \cdots$$

$$\cdots + \frac{h}{3}\left(y_{N-2} + 4y_{N-1} + y_N\right).$$

$$\therefore S_N = \frac{h}{3}\left[y_0 + y_N + 4\sum_{\substack{0<k<N \\ \text{odd}}} y_k + 2\sum_{\substack{0<k<N \\ \text{even}}} y_k \right]$$