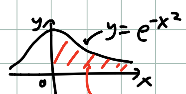
Lec 16

Estimating convergence/divergence of improper integrals §6.5

※ For improper integrals' convergence, what matters is behaviour of the function near singularities or near $\pm\infty$.

- Try comparism with some other functions whose decay/growth behaviour is EASIER to see.

e.g. Is $\int_0^\infty e^{-x^2}\,dx$ convergent?



Is the area make sense?

$\int_0^R e^{-x^2}\,dx \leftarrow \int e^{-x^2}\,dx$ is something we cannot explicitly compute. unfortunately

Estimate by comparing with a SIMPLER function!

- Observe // e^{-x^2} decays fast as $x \to \infty$

e.g. faster than e^{-x}. //

Namely, $e^{-x^2} \le e^{-x}$ for $x \geq 1$. $\leftarrow$ $\boxed{\text{Comparison.}}$

$$\therefore \int_1^R e^{-x^2}\,dx \le \int_1^R e^{-x}\,dx = -e^{-R} + e^{-1} \le e^{-1}.$$

① Thus $I_R := \int_1^R e^{-x^2}\,dx \le e^{-1}$

the point is $\int_1^R e^{-x^2}\,dx$ is bounded.

- ② Note I_R is increasing in R

(This is intuitively obvious since I_R = area the bigger R, the larger –

This can be seen since.

$\underbrace{\frac{d}{dR} I_R = e^{-R^2} > 0}$. $\therefore$ increasing in R

This works for $I_R = \int_a^R f(t)\,dt$ if $f(R) > 0$ for all R.

What is use from ① is the boundedness and the value e is not the point.

① & ② implies (see Next page for corresponding theorem & proof)

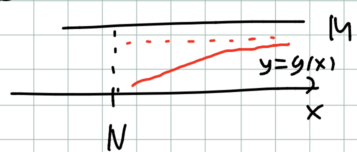
- $\lim_{R\to\infty} I_R$ exists
- $\lim_{R\to\infty} I_R \le e^{-1}$.

Finally
$$\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$$
$$= \int_0^1 e^{-x^2}dx + \underbrace{\lim_{R\to\infty}\int_1^R e^{-x^2}dx}_{\text{converges. to a finite number } \le e.}$$

So, $\int_0^\infty e^{-x^2}dx$ converges, to a finite number $\le e$. ▨

- Thm If $g(x)$ is monotonically increasing in x and $g(x) \le M$ for all x (or for all $x \ge N$ some fixed N)

then $\lim_{x\to\infty} g(x)$ exists & $\lim_{x\to\infty} g(x) \le M$.



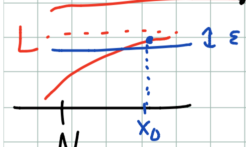
proof

Your exercise.

Let $L = \sup\{g(x) \mid x \ge N\}$. ⟵ We will see $L = \lim_{x\to\infty} g(x)$.

Note $L \le M$.

Fix $\varepsilon > 0$. Choose any $x_0 > N$, such that $g(x_0) > L - \varepsilon$



Then since g is assumed to be monotonically increasing, for all $x \geq x_0$ $\quad g(x) \geq g(x_0)$

$* \cdots$ thus for all $x > x_0$,
$g(x) \geq L - \varepsilon$

$** \cdots$ On the other hand
$\forall x \geq N$, $\quad g(x) \leq L$ (by definition of L)

Combining $*$ & $**$, we have $\quad |g(x) - L| \leq \varepsilon$ for all $x \geq x_0$.

This means $\lim_{x\to\infty} g(x) = L$ $\quad$ The limit exists & its value is L. $\quad$ Note $L \leq M$. ▨

Similarly, we can show:

- <u>Thm</u> If $g(x)$ is monotonically decreasing in x
and $g(x) \geq M$ for all x (or for all $x \geq N$ some fixed N)
then $\lim_{x\to\infty} g(x)$ exists & $\lim_{x\to\infty} g(x) \geq M$.

The above results & method
can be used to show the following easily:

- <u>Thm</u> Comparison test for improper integrals

$-\infty \leq a \leq < b \leq \infty$. $\quad 0 \leq f(x) \leq g(x)$

If $\int_a^b f(x)dx$ diverges then $\int_a^b g(x)dx$ diverges

If $\int_a^b g(x)dx$ converges then $\int_a^b f(x)dx$ converges. & $\int_a^b f(x)dx \leq \int_a^b g(x)dx$

<u>proof</u> your exercise.

Here the assumption $0 \leq f \leq g$ was crucial
since it implies monotone increase of the integrals as the intervals get larger.

see §6.5. Thm 3. for the proof. ▨

More comparisons. Guess & Show!

e.g. p-integrals.

For $p>0$,

- $\int_0^1 \frac{1}{x^p}\,dx = \lim_{c\to 0+} \int_c^1 \frac{1}{x^p}\,dx = \begin{cases} \text{Case } p=1: \quad \lim_{c\to 0^+} \int_c^1 \frac{1}{x}\,dx = \lim_{c\to 0+} \left[\ln|x|\right]_c^1 = +\infty. \\ \text{Case } p\neq 1 \\ \lim_{c\to 0+} \left[\frac{x^{1-p}}{1-p}\right]_c^1 = \lim_{c\to 0+}\left[\frac{1}{1-p} - \frac{1}{1-p}c^{1-p}\right] \\ = \begin{cases} +\infty & \text{if } p>1 \\ \frac{1}{1-p} & \text{if } 0<p<1 \end{cases} \end{cases}$

- Similarly $\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} +\infty & \text{if } 0<p\le 1 \\ \frac{1}{p-1} & \text{if } p>1 \end{cases}$

Can use these facts to handle:

EX Converge / diverge? Guess & Show!

- $\int_1^\infty \frac{\sqrt{x}}{\sqrt{x}+x^2}\,dx$:

$\frac{\sqrt{x}}{\sqrt{x}+x^2}$ for very large x, (notation: $x \gg 1$) "behaves like" $\frac{\sqrt{x}}{x^2} = \frac{1}{x^{\frac{3}{2}}}$ since $\sqrt{x} \ll x^2$ for $x \gg 1$

But $\int_1^\infty \frac{1}{x^{\frac{3}{2}}}\,dx$ converges. So, we can guess convergence of $\int_1^\infty \frac{\sqrt{x}}{\sqrt{x}+x^2}\,dx$.

To show this rigorously,

Note $\frac{\sqrt{x}}{\sqrt{x}+x^2} \le \frac{1}{1+x^{\frac{3}{2}}} \le \frac{1}{x^{\frac{3}{2}}}$ for $x>0$

Therefore, $\int_1^\infty \frac{\sqrt{x}}{\sqrt{x}+x^2}\,dx \le \int_1^\infty \frac{1}{x^{\frac{3}{2}}}\,dx \overset{\text{finite!}}{<} \infty$

Thus, $\int_1^\infty \frac{\sqrt{x}}{\sqrt{x}+x^2}\,dx$ converges due to the comparison thm.

- $\int_0^\pi \frac{\sin x}{x^2}\,dx$ singularity at $x=0$.

Guess: as $x\to 0$, $\sin x$ behaves like x (why since $\lim_{x\to 0}\frac{\sin x}{x} = 1$, $\sin x \approx x$ for $x\approx 0$)

$\therefore \frac{\sin x}{x^2} \overset{\text{behaves like}}{\sim} \frac{x}{x^2} = \frac{1}{x}$ for $|x| \ll 1$.

But $\int_0^\pi \frac{1}{x}\,dx = \infty$. So can guess $\int_0^\pi \frac{\sin x}{x^2}\,dx$ diverges.

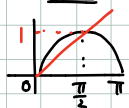
To prove this rigorously let us try $\boxed{\sin x \le x \text{ for } x \ge 0}$ $\Longleftarrow$ Note: $\sin 0 = 0$

$(\sin x)' = \cos x \le 1 = (x)'$

$\therefore \ \sin x = \int_0^x \cos t \, dt \le \int_0^x 1 \, dt = x$ for $x \ge 0$.

$$\therefore \int_0^{\pi} \frac{\sin x}{x^2} dx \le \int_0^{\pi} \frac{x}{x^2} dx = +\infty.$$

But, this does not help to conclude anyting.

Instead, can consider 

$\sin x \ge \frac{2}{\pi} x$ for $0 \le x \le \frac{\pi}{2}$

$$\therefore \int_0^{\pi} \frac{\sin x}{x^2} dx = \underbrace{\int_0^{\frac{\pi}{2}} \frac{\sin x}{x^2} dx}_{\ge \int_0^{\frac{\pi}{2}} \frac{2}{\pi} \frac{x}{x^2} dx = +\infty.} + \underbrace{\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{x^2} dx}_{\text{proper integral, no singularity}},$$

proper integral, no singularity, so, this does NOT affect convergence/divergence.

$\int_0^1 \frac{1}{x} dx = +\infty$

Thus, from comparisan theorem, can conclude the integral $\int_0^{\pi} \frac{\sin x}{x^2} dx$ diverges.

- $$\int_1^{\infty} \frac{e^{x^2}}{x + e^{x^3}} dx \qquad \frac{e^{x^2}}{x + e^{x^3}} = \frac{1}{xe^{-x^2} + e^{x}} \le \frac{1}{e^x} \quad \text{for } x > 0$$

$$\le \int_1^{\infty} e^{-x} dx < +\infty$$

$\therefore$ converges. due to the comparisan thm.