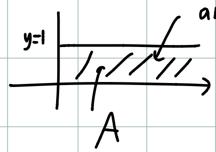
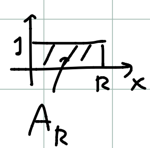
Lec 15 Improper Integrals. § 6.5 ← Read!

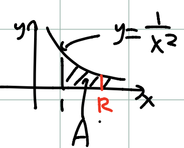


area of the unbounded region "=" lim (area of bounded region)



$$\text{Area of } A = \lim_{R\to\infty} A_R = \lim_{R\to\infty} R = \infty.$$

e.g.

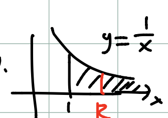


$$\text{Area of } A = \lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx$$

$$= \lim_{R\to\infty} \left[-x^{-1}\right]_1^R$$

$$= \lim_{R\to\infty} \left(-R^{-1}+1\right) = 0+1 = \underline{1}$$

e.g.



$$\text{area} = \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx$$

$$= \lim_{R\to\infty} \left[\ln x\right]_1^R = \lim_{R\to\infty} \left(\ln R - \ln 1\right) = +\infty.$$

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{R\to\infty} \int_{-R}^b f(x)\,dx$$

← If the limit exists, the integral is said to "converges" otherwise, "diverges".

If the limit is $+\infty$ $(-\infty)$ then said to "diverges to $+\infty$ $(-\infty)$"

e.g. $\int_0^\infty \cos x\,dx = \lim_{R\to\infty} \int_0^R \cos x\,dx$

$= \lim_{R\to\infty} \left[\cos x\right]_0^R = \lim_{R\to\infty} \left[\cos R - 1\right]$ ← Such limit does <u>not</u> exist.

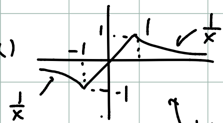
Note

$$\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$$

but, $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

For this to converge, both of these must converges.

e.g. $g(x)$



odd function

$$\int_{-\infty}^{\infty} g(x)\,dx = \lim_{R\to\infty}\underbrace{\int_{-R}^{R} g(x)\,dx}_{=0 \text{ since } g \text{ is odd}} = 0$$

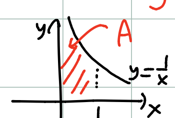
WRONG

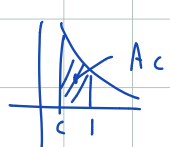
$$\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{0} g(x)\,dx + \int_{0}^{\infty} g(x)\,dx$$

does NOT converge   does not converge

$\therefore \int_{-\infty}^{\infty} g(x)\,dx$ does NOT converge.

e.g. "unbounded region" ← the value of the function tends to $\pm\infty$.

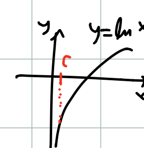




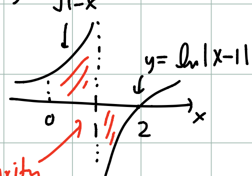
$$\text{Area } A = \lim_{c\to 0+}(\text{Area of } A_c)$$

$$= \lim_{c\to 0+}\int_{c}^{1}\frac{1}{x}\,dx = \lim_{c\to 0+}\Big[\ln|x|\Big]_{c}^{1} = \lim_{c\to 0+}(\cancel{\ln 1}^{0} - \ln c)$$

$$= \lim_{c\to 0+}(-\ln c) = +\infty$$



e.g. $y=\frac{1}{\sqrt{1-x}}$



singularity

$$f(x)=\begin{cases}\frac{1}{\sqrt{1-x}} & x<1\\ \ln|x-1| & x>1\end{cases}$$

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \lim_{c\to 1^-} \int_0^c \frac{1}{\sqrt{1-x}}\,dx + \lim_{c\to 1^+} \int_c^2 \ln|x-1|\,dx$$

$u = x-1$

$\int \ln u\,du = u\ln u - \int u\cdot\frac{1}{u}\,du$

$= u\ln u - u + C$

$$= \lim_{c\to 1^-} \Big[-2\sqrt{1-x}\Big]_0^c + \lim_{c\to 1^+} \Big[(x-1)\ln(x-1) - (x-1)\Big]_c^2$$

$$= \lim_{c\to 1^-}\left(-2\sqrt{1-c} + 2\sqrt{1}\right) + \lim_{c\to 1^+}\left[\cancel{\ln 1}^{\,0} - 1 - (c-1)\ln(c-1) + c-1\right]$$

$$= \left(-2\cdot\cancel{0}^{\,0} + 2\right) + -1$$

$$-\underbrace{\lim_{c\to 1^+}(c-1)\ln(c-1)}_{=0.} + 1 - 1.$$

$$\lim_{c\to 1^+}(c-1)\ln(c-1) = \lim_{u\to 0^+} u\ln u = \lim_{u\to 0^+}\frac{\ln u}{\frac{1}{u}} = \lim_{u\to 0^+}\frac{\frac{1}{u}}{-\frac{1}{u^2}} = \lim_{u\to 0^+}(-u) = 0$$

L'Hopital.

$$\therefore \int_0^2 f(x)\,dx = \underline{\underline{1}}. \quad \boxed{/\!/}$$

· F.T.C. for improper integrals.

Suppose $\int_{-\infty}^a f(t)\,dt$ converges for some $a \in \mathbb{R}$ & f is continuous on $\mathbb{R}$

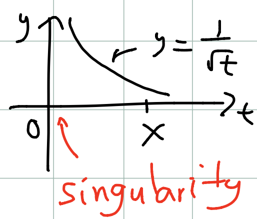
Then $$\frac{d}{dx}\int_{-\infty}^x f(t)\,dt = f(x)$$

(reason) $$\int_{-\infty}^x f(t)\,dt = \underbrace{\int_{-\infty}^a f(t)\,dt}_{\text{constant!}} + \int_a^x f(t)\,dt.$$

$\therefore$ $\boxed{/\!/}$

Ex For $x > 0$, $\dfrac{d}{dx}\displaystyle\int_0^x \frac{1}{\sqrt{t}}\,dt = ?$

<sol>



$$\int_0^x \frac{1}{\sqrt{t}}\,dt = \underbrace{\int_0^1 \frac{1}{\sqrt{t}}\,dt}_{\text{constant}} + \int_1^x \frac{1}{\sqrt{t}}\,dt$$

since the improper integral converges.

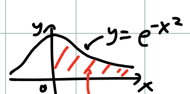
$$\therefore \frac{d}{dx}\int_0^x \frac{1}{\sqrt{t}}\,dt = \frac{d}{dx}\int_1^x \frac{1}{\sqrt{t}}\,dt = \frac{1}{\sqrt{x}}. \quad \square$$

this works as long as $x > 0$. since for $0 < x \le t \le 1$, $\frac{1}{\sqrt{t}}$ is continuous.

Estimating convergence/divergence of improper integrals. §6.5

: comparison.

e.g. Is $\int_0^\infty e^{-x^2}\,dx$ convergent?



Is the area make sense?

$$\int_0^R e^{-x^2}\,dx \leftarrow$$ unfortunately $\int e^{-x^2}dx$ is something we cannot explicitly compute.

Estimate by comparing with a SIMPLER function!