Lec14 §6.3. Inverse substitution.

Ex. $\int \frac{dx}{\sqrt{x^2-2x}}$ ← due to $\sqrt{\phantom{x}}$ partial fraction metod does NOT work

$= \int \frac{dx}{\sqrt{x^2-2x+1-1}}$

$= \int \frac{dx}{\sqrt{(x-1)^2-1}}$

$= \int \frac{du}{\sqrt{u^2-1}}$ ← $u = x-1$. $u^2-1>0$

This makes the integral in a SIMPLER form.

How to compute $\int \frac{du}{\sqrt{u^2-1}}$ ?

Let $u = \sec\theta$ $\quad du = \left(\frac{1}{\cos\theta}\right)' d\theta = \frac{\sin\theta}{\cos^2\theta} d\theta$

$u>1$ or $u<-1$.

$u^2-1 = \sec^2\theta - 1 = \tan^2\theta$

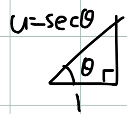
$\therefore \sqrt{u^2-1} = \sqrt{\tan^2\theta} = |\tan\theta|$.

$\therefore \int \frac{du}{\sqrt{u^2-1}} = \int \frac{1}{|\tan\theta|} \cdot \frac{\sin\theta}{\cos^2\theta} d\theta$

Case $\tan\theta > 0$ : ($u>1$) $\quad \int \frac{du}{\sqrt{u^2-1}} = \int \frac{1}{\tan\theta} \frac{\sin\theta}{\cos^2\theta} d\theta = \int \frac{d\theta}{\cos\theta} = \ln|\sec\theta+\tan\theta| + C$

$\int \sec\theta \, d\theta$ ↓

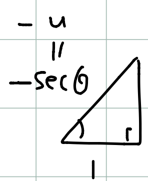
don't have to memorize this.




$\tan\theta = \sqrt{u^2-1}$ $\quad = \ln\left| u + \sqrt{u^2-1} \right| + C$

<u>Case</u> $\tan\theta < 0$: $\quad \displaystyle\int \frac{du}{\sqrt{u^2-1}} = -\int \frac{d\theta}{\cos\theta} = -\ln|\sec\theta + \tan\theta| + C.$

$(u = \sec\theta < -1)$




$-\tan\theta = \sqrt{u^2-1}$

since $\tan\theta < 0$.

$$= -\ln\left| u + -\sqrt{u^2-1} \right| + C$$

$$= -\ln\left| u - \sqrt{u^2-1} \right| + C$$

<u>Note</u> $-\ln\left|u - \sqrt{u^2-1}\right|$

$$= \ln\left|\frac{1}{u-\sqrt{u^2-1}}\right| = \ln\left|\frac{1}{u-\sqrt{u^2-1}} \cdot \frac{u+\sqrt{u^2-1}}{u+\sqrt{u^2-1}}\right|$$

$$= \ln\left|\frac{u+\sqrt{u^2-1}}{u^2-u^2+1}\right| = \ln\left|u+\sqrt{u^2-1}\right|$$

$$\therefore \quad \int \frac{du}{\sqrt{u^2-1}} = \ln\left|u + \sqrt{u^2-1}\right| + C$$

- Another method: cosh, sinh, tanh

$\cosh(x) = \dfrac{e^x + e^{-x}}{2} \geq 1$ show.! can cover all $y \geq 1$.

$\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ can cover all $y \in \mathbb{R}$

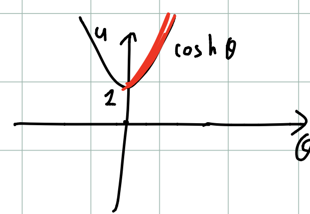
$\sinh(x) > 0$ if $x > 0$

$\sinh(x) < 0$ if $x < 0$.

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, $|\tanh(x)| < 1$

- $\cosh^2 x - \sinh^2 x = 1$
- $\sinh^2 x = \cosh^2 x - 1$
- $\cosh'(x) = \sinh(x)$

$\sinh'(x) = \cosh(x)$




EX. $\int \frac{du}{\sqrt{u^2-1}}$ ← $|u|>1$

Case: $u>1$

$u = \cosh\theta \quad du = \sinh\theta\, d\theta$

this substitution can cover only $u \geq 1$.

choose $\theta \geq 0$.

This still covers all $u \geq 1$.

$$= \int \frac{1}{\sqrt{\sinh^2\theta}} \sinh\theta\, d\theta$$

$$= \int \frac{\sinh\theta}{|\sinh\theta|} d\theta$$

$$= \int \frac{\sinh\theta}{\sinh\theta} d\theta$$

for $\theta \geq 0$ $\sinh\theta \geq 0$.

$u^2 - 1 = \cosh^2\theta - 1 = \sinh^2\theta$

$$= \int d\theta = \theta + C$$

$$= \cosh^{-1}(u) + C$$

Case $u < -1$: choose $u = -\cosh\theta$ $\theta \geq 0$, $du = -\sinh\theta\, d\theta$

Then this covers all $u \leq -1$.

$$\therefore \int \frac{du}{\sqrt{u^2-1}} = \int \frac{-\sinh\theta\, d\theta}{\sinh\theta}$$

for $\theta \geq 0$ $\sinh\theta > 0$

$u = -\cosh\theta$

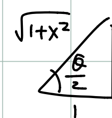
$$= -\int d\theta = -\theta + C = -\cosh^{-1}(-u) + C.$$

- Rational functions of $\sin\theta$, $\cos\theta$

<u>Ex</u>. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\cos\theta+\sin\theta}$

Try:

Technique: $x = \tan\frac{\theta}{2}$ ← half angle.




$x = \tan\frac{\theta}{2}$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{1}{1+x^2} - \frac{x^2}{1+x^2} = \frac{1-x^2}{1+x^2}$$

$$\sin\theta = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2} = 2\,\frac{1}{\sqrt{1+x^2}}\cdot\frac{x}{\sqrt{1+x^2}} = \frac{2x}{1+x^2}$$

$$dx = \left(\tan\frac{\theta}{2}\right)' d\theta = \sec^2\frac{\theta}{2}\cdot\frac{1}{2}d\theta$$

$$= (1+x^2)\,\frac{1}{2}d\theta$$

$$\therefore\ d\theta = \frac{2}{1+x^2}dx.$$

**POINT**

All these are rational functions of $x$

So. $\left(\begin{array}{c}\text{rational functions}\\ \text{of } \cos\theta,\ \sin\theta\end{array}\right) d\theta$

$= \begin{array}{c}\text{rational functions}\\ \text{of } x\end{array} \cdot dx$ for $x = \tan\frac{\theta}{2}$

So. $$\int\frac{d\theta}{1+\cos\theta+\sin\theta} = \int\frac{1}{1+\frac{1-x^2}{1+x^2}+\frac{2x}{1+x^2}}\ \frac{2\,dx}{1+x^2}$$

$$= \int\frac{1}{1+x^2+1-x^2+2x}\cdot 2\,dx$$

$$= \int\frac{1}{1+x}\,dx = \ln|1+x| + C.$$

$$= \ln\left|1+\tan\frac{\theta}{2}\right| + C$$