Lec 6

- one more example of Riemann integrability §5.3 & Appendix IV
- properties of the definite integral §5.4

Q Is every bound function in $[a,b]$ integrable?

Ans No! e.g. $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

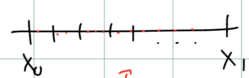
Appendix IV Ex. 2.

is <u>not</u> integrable on any $[a,b]$, in particular $[0,1]$.

proof For every partition $P: 0 = x_0 < x_1 < \cdots < x_N = 1$,

each subinterval $[x_{i-1}, x_i]$

contains both rational & irrational numbers.



both rational numbers / irrational numbers are dense

Thus $M_i = 1$, $m_i = 0$.

$$\therefore \quad U(f,P) = \sum_{i=1}^{N} M_i \Delta x_i = \sum_{i=1}^{N} 1 \cdot \Delta x_i = 1$$

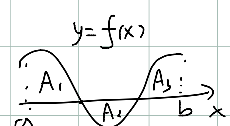
$$L(f,P) = \sum_{i=1}^{N} m_i \Delta x_i = 0.$$

They cannot be within $\varepsilon = \frac{1}{2}$ ☒

Properties of the definite integral

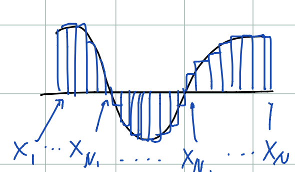
Observe Integral as <u>signed areas</u>

e.g. for illustration



A_i: areas.

$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3.$$



$$\int_a^b f(x)\,dx \overset{\text{approx}}{\approx} \sum_{k=1}^{N} f(x_k^*)\Delta x_k$$

$$= \underbrace{\sum_{k=1}^{N_1} \overbrace{f(x_k^*)}^{>0} \Delta x_k}_{\approx A_1} + \underbrace{\sum_{k=N_1}^{N_2} \overbrace{f(x_k^*)}^{<0} \Delta x_k}_{\approx -A_2} + \underbrace{\sum_{k=N_2}^{N} \overbrace{f(x_k^*)}^{>0} \Delta x_k}_{\approx A_3}.$$

Partition:

$a = x_0 < x_1 < \cdots < x_{N_1} < \cdots < x_{N_2} < \cdots < x_N = b$

This idea "$\int_a^b f(x)\,dx$ = Signed area"

works well for all Riemann integrable functions.

and it leads to the following properties.

(We skip the rigorous proofs)

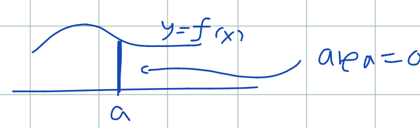
Thm (Properties of the definite integral) [Thm 3. §5.4]

Let f, g be (bounded) & integrable on an interval $[a_0, b_0]$

(Note in this case f, g are integrable on <u>any</u> subinterval, say, $[\alpha, \beta] \subset [a_0, b_0]$)

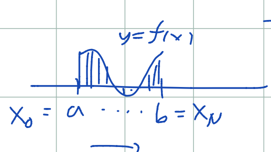
Let $a, b, c \in [a_0, b_0]$.

Then.

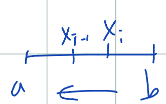


(a) $\int_a^a f(x)\,dx = 0$.

(b) For $a<b$, define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$



Then $\int_a^b f(x)\,dx \overset{\text{approx.}}{\approx} \sum_{i=1}^{N} f(x_i^*)\,(x_i - x_{i-1})$ $\quad \Delta x_i = x_i - x_{i-1}$



$\boxed{\int_b^a} f(x)\,dx \approx \sum_{i=1}^{N} f(x_i^*)\,\underbrace{(x_{i-1} - x_i)}$

opposite order

Signed length. the opposite sign to $(x_i - x_{i-1})$

So, from now on, when we consider $\int_a^b f(x)\,dx$, we do <u>NOT</u> have to assume $a<b$.

(c) <u>linearity</u> $\quad C_1, C_2 \in \mathbb{R}$ constants

$$\int_a^b [C_1 f(x) + C_2 g(x)]\,dx$$

$$= C_1 \int_a^b f(x)\,dx + C_2 \int_a^b g(x)\,dx$$

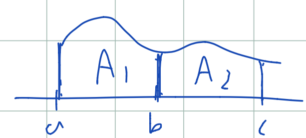
NOTE: This linearity comes from linearity of summation

$$\sum_{k=m}^{n}(c_1 a_k + c_2 b_k) = c_1 \sum_{k=m}^{n} a_k + c_2 \sum_{k=m}^{n} b_k$$

Notice

$$\int_a^b c_1 f(x) + c_2 g(x)\,dx \underset{\text{approx}}{\approx} \sum_{k=1}^{N}\left[c_1 f(x_k^*) + c_2 g(x_k^*)\right]\Delta x_k$$

(d) $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

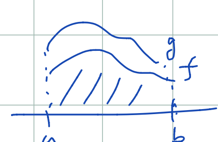
$\because$ 

total area $A = \int_a^c f(x)\,dx$

$A_1 = \int_a^b f(x)\,dx$, $A_2 = \int_b^c f(x)\,dx$

(e) Suppose $\underline{a \le b}$. $f(x) \le g(x)$. on $[a,b]$

Then $$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

Reason 

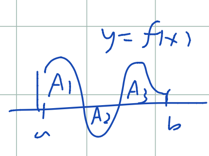
$$\sum_{i=1}^{N} f(x_i^*)\Delta x_i \le \sum_{i=1}^{N} g(x_i^*)\Delta x_i$$

$\Delta x_i \geq 0$ because $a \le b$.

for $f \le g$.

(f) Suppose $a \le b$.

Then, $$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$ (because $-|f(x)| \le f(x) \le |f(x)|$, can apply (e).)

e.g. 

areas A_i.

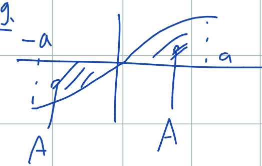
$$\left|\int_a^b f(x)\,dx\right| = |A_1 - A_2 + A_3|$$

$$\le A_1 + A_2 + A_3 = \int_a^b |f(x)|\,dx$$

(g) odd function f. $-f(x) = f(-x)$. $\forall x$.

Then, $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

e.g.

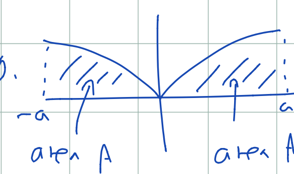


$$\int_{-a}^{a} f(x)\,dx = -A + A = 0.$$

(h) even function f $f(x) = f(-x)$

Then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

e.g.



$$\int_{-a}^{a} f(x)\,dx = A + A$$

$$= 2\int_{0}^{a} f(x)\,dx.$$

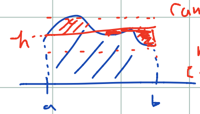
Mean Value thm.

f continuous on $[a,b]$ ← $a<b$.

$\Rightarrow \exists$ $c \in [a,b]$ such that $\int_a^b f(x)\,dx = (b-a)f(c)$.

"There exists"

"reason" Case $f \geq 0$:



can find height h

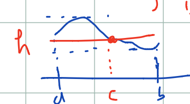
$\min_{[a,b]} f \leq h \leq \max_{[a,b]} f$

such that the area above h is the same as the area of the region below h and above $y=f(x)$

Then $\int_a^b f(x)\,dx = h\cdot(b-a)$.

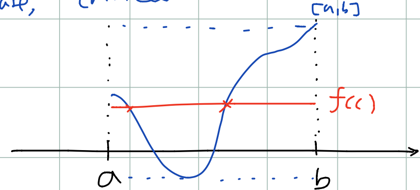
And since $\min_{[a,b]} f(x) \leq h \leq \max_{[a,b]} f(x)$

f is continuous on $[a,b]$



there exists $c \in [a,b]$ s.t. $f(c) = h$.

For general case, consider $f(x) - \min_{[a,b]} f(x) \geq 0$.



More rigorous proof

Let $m = \min_{[a,b]} f(x)$, $M = \max_{[a,b]} f(x)$

By continuity of f on $[a,b]$, such max & min exist, and there are points $\alpha, \beta \in [a,b]$ such that $f(\alpha)=M$, $f(\beta)=m$.

Now, since $m \leq f(x) \leq M$ on $[a,b]$, $a<b$,

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx \Rightarrow m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

$$\Rightarrow m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M.$$

Due to continuity of f & $M = f(\alpha)$, $m = f(\beta)$, we can apply intermediate value theorem and see there exists $c \in [\alpha, \beta]$ if $\alpha \le \beta$ (or $c \in [\beta, \alpha]$ if $\alpha \ge \beta$)

such that $f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$. $\boxed{//}$

note In the mean value theorem, Continuity of f is essential.

Example $f(x) = \begin{cases} 1 & x \le \frac{1}{2} \\ 0 & x > \frac{1}{2}. \end{cases}$ on $[0,1]$.

$$\int_0^1 f(x)\,dx = \frac{1}{2} \neq (1-0) f(c) \quad \text{for any } c \in [0,1].$$