Lec 5 · Riemann integrability §5.3, Appendix IV
· Properties of Rieman integrals §5.4

Def A bounded function f on $[a,b]$ — not necessarily $f \geq 0$.
is said to be Riemann integrable (or simply integrable)
if the following holds:
$\forall \varepsilon > 0$, there exists a partition P of $[a,b]$
such that $U(f,P) - L(f,P) \leq \varepsilon$.

Def (Definite integral)
For bounded Riemann integrable f on $[a,b]$,

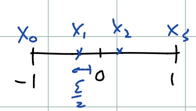
$$\underbrace{\int_a^b f(x)\,dx}_{\text{definition of } \int_a^b f(x)dx \text{ when } f \text{ is integrable on } [a,b]} = \sup_{\substack{P \\ \text{partition of } [a,b]}} L(f,P) = \inf_{\substack{P \\ \text{partition of } [a,b]}} U(f,P).$$

Note: ⊛ ... $L(f,P) \leq \int_a^b f(x)\,dx \leq U(f,P')$ ...

for any partition P, P' of $[a,b]$.

e.g. $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$ on $[-1,1]$

Riemann integrable: $\forall \varepsilon > 0$, choose a partition P: $x_0 = -1$, $x_1 = -\frac{\varepsilon}{2}$, $x_2 = \frac{\varepsilon}{2}$, $x_3 = 1$.



Then $U(f,P)$
$$= \sum_{i=1}^{3} M_i \Delta x_i = 0 \cdot \Delta x_1 + 1 \cdot \Delta x_2 + 0 \, \Delta x_3$$
$$= \Delta x_2 = x_2 - x_1 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$L(f,P) = 0$

$\therefore U(f,P) - L(f,P) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon$.

$\int_{-1}^1 f(x)\,dx = 0$:

Observe $L(f,P) \leq \int_{-1}^1 f(x)\,dx \leq U(f,P')$ for any partitions.

$\forall \varepsilon > 0$, Use the partition: P: $x_0 = -1$, $x_1 = -\frac{\varepsilon}{2}$, $x_2 = \frac{\varepsilon}{2}$, $x_3 = 1$,

then $L(f,P) = 0 \leq \int_{-1}^1 f(x)\,dx \leq U(f,P) = \varepsilon$

Since ε can be arbitrary small, $\int_{-1}^1 f(x)\,dx = 0$ ∎

Many functions are integrable, especially,

Thm If f is continuous on $[a,b]$

Important! then f is integrable on $[a,b]$.

pf Your exercise:

The proof is similar to the warm-up discussion in Lec3.

It uses uniform continuity of f on $[a,b]$.

For the proof,

see Thm5, Appendix IV. ▨

Thm For f bounded & Riemann integrable on $[a,b]$,

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_i^*)\,\Delta x_i \,, \qquad x_i^* \in [x_{i-1}, x_i]$$

for any partition $P: a = x_0 < x_1 < \cdots < x_N = b$ with $\|P\| \to 0$ as $N \to \infty$.

Here, $\|P\| :\overset{\text{def.}}{=} \max_{i=1,\dots,N} |\Delta x_i|$ the mesh size

pf We skip the proof for the general case.

Your exercise

(a) Prove the theorem for the special case where f is continuous

Hint: Step1. Show that it is sufficient to show

$$(*)\cdots\left[\begin{array}{l}\forall \varepsilon>0, \text{ there exist } \delta>0 \text{ such that} \\ \quad \forall \text{ partition } P \text{ with } \|P\|<\delta, \text{ it holds } U(f,P)-L(f,P)<\varepsilon\end{array}\right].$$

Hint: use $L(f,P) \le \int_a^b f(x)\,dx \le U(f,P')$ for any partition P, P' of $[a,b]$.

Step2. Show $(*)$ for the case f is continuous. ▨

(b)* Prove the theorem for any Riemann integrable f on $[a,b]$.

Q Is every bound function in $[a,b]$ integrable ?

Ans No! e.g. $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

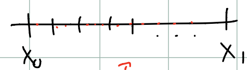
Appendix IV
Ex. 2.

is not integrable on any $[a,b]$, in particular $[0,1]$.

proof For every partition P: $0 = x_0 < x_1 < \cdots < x_N = 1$.

each subinterval $[x_{i-1}, x_i]$

contains both rational & irrational numbers.



both rational numbers / irrational numbers are dense

Thus $M_i = 1$, $m_i = 0$.

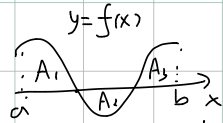
$$\therefore \quad U(f, P) = \sum_{i=1}^{N} M_i \, \Delta x_i = \sum_{i=1}^{N} 1 \cdot \Delta x_i = 1$$

$$L(f, P) = \sum_{i=1}^{N} m_i \, \Delta x_i = 0.$$

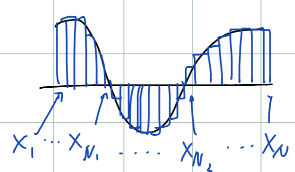
They cannot be within $\varepsilon = \frac{1}{2}$ $\boxtimes$

Observe Integral as signed areas

e.g. for illustration



A_i: areas. $\int_a^b f(x)\,dx = A_1 - A_2 + A_3$.



$$\int_a^b f(x)\,dx \overset{\text{approx}}{\approx} \sum_{k=1}^{N} f(x_k^*)\Delta x_k$$

$$= \underbrace{\sum_{k=1}^{N_1} \overset{>0}{f(x_k^*)}\Delta x_k}_{\approx A_1} + \underbrace{\sum_{k=N_1}^{N_2} \overset{<0}{f(x_k^*)}\Delta x_k}_{\approx -A_2} + \underbrace{\sum_{k=N_2}^{N} \overset{>0}{f(x_k^*)}\Delta x_k}_{\approx A_3}.$$

Partition: $a = x_0 < x_1 < \cdots < x_{N_1} < \cdots < x_{N_2} < \cdots < x_N = b$

This idea "$\int_a^b f(x)\,dx$ = signed area"

works well for all Riemann integrable functions.

and it leads to the following properties.

(We skip the rigorous proofs)

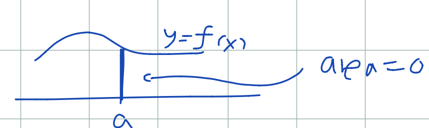
<u>Thm</u> (Properties of the definite integral) [Thm 3. §5.4]

Let f, g be (bounded) & integrable on an interval $[a_0, b_0]$

(Note in this case f, g are integrable on <u>any</u> subinterval, say, $[\alpha, \beta] \subset [a_0, b_0]$)

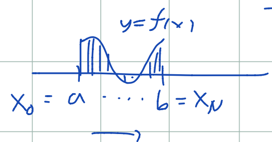
Let $a, b, c \in [a_0, b_0]$.

Then

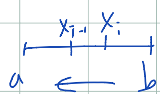


(a) $\int_a^a f(x)\,dx = 0$.

(b) For $a<b$, define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$



Then $\int_a^b f(x)dx \overset{\text{approx.}}{\approx} \sum_{i=1}^{N} f(x_i^*)\underbrace{(x_i - x_{i-1})}_{\Delta x_i = x_i - x_{i-1}}$



$\boxed{\int_b^a} f(x)\,dx \approx \sum_{i=1}^{N} f(x_i^*)\underbrace{(x_{i-1} - x_i)}$

opposite order

signed length. the opposite sign to $(x_i - x_{i-1})$

So, from now on, when we consider

$\int_a^b f(x)dx$, we do <u>NOT</u> have to assume $a<b$.

(c) <u>linearity</u> $c_1, c_2 \in \mathbb{R}$ constants

$$\int_a^b [c_1 f(x) + c_2 g(x)]\,dx$$
$$= c_1\int_a^b f(x)dx + c_2\int_a^b g(x)dx$$

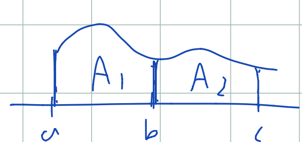
<u>NOTE</u>: This linearity comes from linearity of summation

$$\sum_{k=m}^{n}(c_1 a_k + c_2 b_k) = c_1\sum_{k=m}^{n} a_k + c_2\sum_{k=m}^{n} b_k$$

Notice

$$\int_a^b c_1 f(x) + c_2 g(x)dx \underset{\text{approx}}{\approx} \sum_{k=1}^{N}\left[c_1 f(x_k^*) + c_2 g(x_k^*)\right]\Delta x_k$$

(d) $\int_a^b f(x)\,dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

$\because$ 

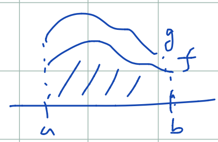
total area $A = \int_a^c f(x)dx$

$A_1 = \int_a^b f(x)dx$, $A_2 = \int_b^c f(x)dx$

(e) Suppose $\underline{a \le b}$. $f(x) \le g(x)$. on $[a,b]$

Then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Reason



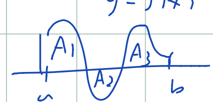
$\sum_{i=1}^{N} f(x_i^*)\,\Delta x_i \le \sum_{i=1}^{N} g(x_i^*)\,\Delta x_i$

for $f \le g$.

(f) Suppose $a \le b$.

Then, $\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx$

e.g.



area A_i.

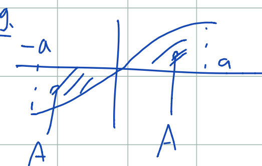
$\left| \int_a^b f(x)\,dx \right| = \left| A_1 - A_2 + A_3 \right|$

$\le A_1 + A_2 + A_3 = \int_a^b |f(x)|\,dx$

(g) odd function f. $-f(x) = f(-x)$. $\forall x$.

Then, $\int_{-a}^{a} f(x)\,dx = 0$

e.g.

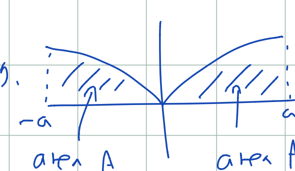


$\int_{-a}^{a} f(x)\,dx = -A + A = 0$.

(h) even function f $f(x) = f(-x)$

Then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

e.g.



$\int_{-a}^{a} f(x)\,dx = A + A$

$= 2\int_0^a f(x)\,dx$.