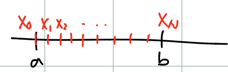
# Lec 4 Riemann integrability : §5.3, Appendix IV.

- Partitions

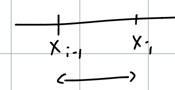
- partition $P$ of $[a, b]$




$$P = \{x_0, x_1, \cdots, x_N\}$$

$$a = x_0 < x_1 < \cdots < x_N = b.$$

i-th subinterval $[x_{i-1}, x_i]$




$\Delta x_i = x_i - x_{i-1}$.

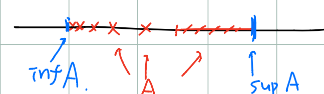
- $[a, b] = \bigcup_{i=1}^{N} [x_{i-1}, x_i]$

- $\sum_{i=1}^{N} \Delta x_i = b - a$.

Note We do not assume $\Delta x_i = \frac{b-a}{N}$

- Upper & lower Riemann sums.

Recall $A \subset \mathbb{R}$. $\sup A$ = least upper bound of $A$, $\inf A$ = greatest lower bound of $A$.

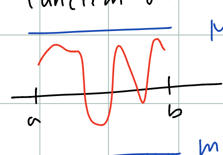



e.g. $A = \{1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots\}$

$\sup A = 1$, $\inf A = 0$.

Note $\sup A$, $\inf A$ may not belong to $A$.

Let $f(x)$ be a bound function on $[a, b]$ : i.e. $\exists$ $m, M$ such that $m \le f(x) \le M$ $\forall x \in [a,b]$.



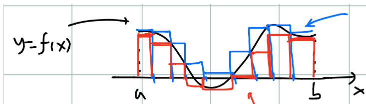

Let $P$: $x_0 = a < x_1 < \cdots < x_N = b$ be a partition of $[a, b]$.

Upper Riemann sum of $f$ corresponding to $P$

$$U(f, P) = \sum_{i=1}^{N} M_i \, \Delta x_i \qquad M_i = \sup_{[x_{i-1}, x_i]} f(x) = \sup\{f(x) \mid x_{i-1} \le x \le x_i\}.$$

Lower Riemann Sum

$$L(f, P) = \sum_{i=1}^{N} m_i \, \Delta x_i \qquad m_i = \inf_{[x_{i-1}, x_i]} f(x) = \inf\{f(x) \mid x_{i-1} \le x \le x_i\}$$




blue rectangles correspond to $U(f,P)$

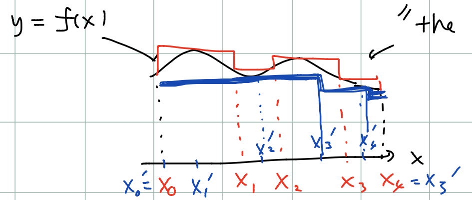
red rectangles correspond to $L(f,P)$

Note: For the same partition $P$, it always holds $L(f,P) \le U(f,P)$

In fact,

**Thm** For any two partitions $P$, $P'$ of $[a,b]$

$$\boxed{L(f,P') \le U(f,P)}$$




"the upper blocks are above the lower blocks, even with different partitions"

$P = \{x_0, x_1, x_2, x_4\}$

$P' = \{x_0', \ldots, x_5'\}$

$x_0' = x_0 \quad x_1' \quad x_1 \quad x_2 \quad x_3 \quad x_4 = x_3'$

For a rigorous proof, try exercise 18 in §5.3.

For full solution, see Thm 1 & 2, Appendix IV.

Note: $L(f,P) \le U(f,P')$ — this inequality can be strict

$$\boxed{\sup_{\substack{P \\ \text{partition} \\ \text{of } [a,b]}} L(f,P) \le \inf_{\substack{P \\ \text{partition} \\ \text{of } [a,b]}} U(f,P)}$$

## Riemann Sums

Given a partition $P: a = x_0 < x_1 < \cdots < x_N = b$ of $[a,b]$.

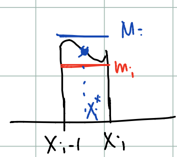
can choose sample points $x_i^* \in [x_{i-1}, x_i]$.

the corresponding Riemann sum is defined as

$$\sum_{i=1}^{N} f(x_i^*) \Delta x_i$$

Note: $L(f,P) \le \sum_{i=1}^{N} f(x_i^*) \Delta x \le U(f,P)$

Reason




$m_i \le f(x_i^*) \le M_i$

$\therefore \quad m_i \Delta x_i \le f(x_i^*) \Delta x_i \le M_i \Delta x_i$

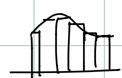
Take sum

$$\underbrace{\sum_{i=1}^{N} m_i \Delta x_i}_{= L(f,P)} \le \sum_{i=1}^{N} f(x_i^*) \Delta x_i \le \underbrace{\sum_{i=1}^{N} M_i \Delta x_i}_{= U(f,P)}$$

▨

• <u>Riemann Integrability</u>

Recall "area as limits of sums".



"$\lim_{N\to\infty} \sum_{i=1}^{N} f(x_i^*)\Delta x_i = \int_a^b f(x)\,dx$"

To make this work we <u>have to</u> make sure

the error of the Riemann sum approximation (i.e. $\sum_{i=1}^{N} f(x_i^*)\Delta x_i$)

gets smaller & smaller as $N\to\infty$
(as the partition gets <u>finer & finer</u> (it means $\Delta x_i \to 0$))

(It was the case, for continuous $f$ on a closed interval $[a,b]$.)

Notice: In this case

the error $= \left| \underbrace{\int_a^b f(x)dx}_{\text{the area.}} - \sum_{i=1}^{N} f(x_i^*)\Delta x_i \right| \le U(f,P) - L(f,P)$

Riemann integrability is a rigorous way of saying this,

using the fact that

the error of Riemann sum approximation $\sum_{i=1}^{N} f(x_i^*)\Delta x_i$

is bounded by

$$U(f,P) - L(f,P)$$

**Def** A bounded function $f$ on $[a,b]$ <u>not</u> necessarily $f \geq 0$.

is said to be <u>Riemann integrable</u> (or simply integrable)

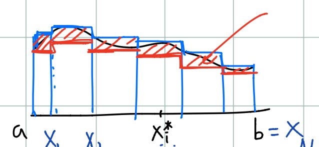
if the following holds:

$\forall\, \varepsilon > 0$, there exists a partition $P$ of $[a,b]$

such that $U(f,P) - L(f,P) < \varepsilon$.

$f$ is integrable on $[a,b] \iff \forall\, \varepsilon > 0$, $\exists$ partition $P$ with

the red area $< \varepsilon$



"bounded $f$ on $[a,b]$ is Riem. integrable
$\Longleftrightarrow$ the error $U(f,P)-L(f,P)$ can be made as small as possible
by choosing an appropriate partition $P$ of $[a,b]$."

For a bounded function $f$ on $[a,b]$

- $f$ is Riemann integrable $\Longleftrightarrow$ $\displaystyle \sup_{\substack{P \\ \text{partition} \\ \text{of } [a,b]}} L(f,P) \overset{\text{No gap!}}{=} \inf_{\substack{P \\ \text{partition} \\ \text{of } [a,b]}} U(f,P)$

- $f$ is not Riemann integrable $\Longleftrightarrow$ $\displaystyle \sup_{\substack{P \\ \text{partition} \\ \text{of } [a,b]}} L(f,P) \underset{\text{gap!}}{<} \inf_{\substack{P \\ \text{partition} \\ \text{of } [a,b]}} U(f,P)$

**Def** (Definite integral)

For bounded Riemann integrable $f$ on $[a,b]$.

$$\int_a^b f(x)\,dx = \sup_{\substack{P \\ \text{partition} \\ \text{of } [a,b]}} L(f,P) = \inf_{\substack{P \\ \text{partition} \\ \text{of } [a,b]}} U(f,P).$$

Note: $\circledast$ ... $L(f,P) \le \int_a^b f(x)\,dx \le U(f,P')$ ...

for any partition $P$, $P'$ of $[a,b]$.

Many functions are integrable, especially,

**Thm** If $f$ is continuous on $[a,b]$
Important! then $f$ is integrable on $[a,b]$.

Pf Your exercise.
The proof is similar to the warm-up discussion in Lec3.
It uses uniform continuity of $f$ on $[a,b]$.

For the proof,
see Thm5, Appendix IV. ▨

**Thm** For $f$ bounded & Riemann integrable on $[a,b]$,

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_i^*)\,\Delta x_i, \qquad x_i^* \in [x_{i-1}, x_i]$$

for any partition $P: a = x_0 < x_1 < \cdots < x_N = b$ with $\|P\| \to 0$ as $N \to \infty$.

Here, $\|P\| \overset{\text{def.}}{=} \max_{i=1,\dots,N} |\Delta x_i|$ the mesh size

**pf** We skip the proof for the general case.

**Your exercise** Prove the theorem for the special case where $f$ is continuous

Hint: Step 1. Show that it is sufficient to show

$$(*) \cdots \left[ \begin{array}{l} \forall \varepsilon > 0, \text{ there exist } \delta > 0 \text{ such that} \\ \quad \forall \text{ partition } P \text{ with } \|P\| < \delta, \text{ it holds } U(f,P) - L(f,P) < \varepsilon \end{array} \right].$$

Hint: use $L(f,P) \le \int_a^b f(x)\,dx \le U(f,P')$ for any partition $P, P'$ of $[a,b]$.

Step 2 Show $(*)$ for the case $f$ is continuous. ▨

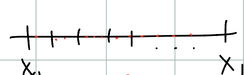
**Q** Is every bound function on $[a,b]$ integrable?

**Ans** No! e.g. $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Appendix IV Ex. 2. is not integrable on any $[a,b]$, in particular $[0,1]$.

**proof** For every partition $P: 0 = x_0 < x_1 < \cdots < x_N = 1$.

each subinterval $[x_{i-1}, x_i]$ contains both rational & irrational numbers.



both rational numbers / irrational numbers are dense

Thus $M_i = 1$, $m_i = 0$.

$$\therefore\ U(f,P) = \sum_{i=1}^{N} M_i\,\Delta x_i = \sum_{i=1}^{N} 1 \cdot \Delta x_i = 1$$

$$L(f,P) = \sum_{i=1}^{N} m_i\,\Delta x_i = 0.$$

They cannot be within $\varepsilon = \frac{1}{2}$ ▨