Lec 3

- § 5.2 two examples.

Announcement about final grades.

- The Definite integral : a warm-up discussion. part of § 5.2 ~ 3.
- Uniform continuity Appendix IV.

※ About 3 lectures from now can be the most difficult abstract lectures to understand during the course.
(about definition of integrals)

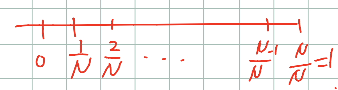
- examples for area as limits of sums.

e.g. $$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{N^2}\sqrt{N^2-k^2} = ?$$

$$\langle sol \rangle = \lim_{N\to\infty} \sum_{k=1}^{N} \sqrt{\left(1-\left(\frac{k}{N}\right)^2\right)N^2}\ \frac{1}{N^2}$$

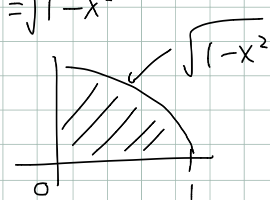
$$\frac{N}{N^2}$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} \underbrace{\sqrt{1-\left(\frac{k}{N}\right)^2}}_{f\left(\frac{k}{N}\right)} \cdot \underbrace{\frac{1}{N}}_{\Delta x_k}$$



$$f(x) = \sqrt{1-x^2}$$

= area of

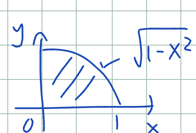


$$= \int_0^1 \sqrt{1-x^2}\,dx.$$

$$= \frac{\pi}{4}$$

Here, we had.

$$\lim_{N\to\infty} \sum_{k=1}^{N} \sqrt{1-\left(\frac{k}{N}\right)^2}\ \frac{1}{N} = \int_0^1 \sqrt{1-x^2}\ dx$$



Observe the correspondence of the notation

$$\lim_{N\to\infty} \sum_{k=1}^{N} \longrightarrow \int_0^1$$

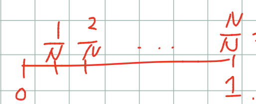
$$\frac{k}{N} \longrightarrow x$$

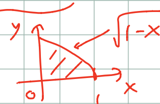
$$\frac{1}{N} \longrightarrow dx$$

e.g. $$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{N}\sqrt{N-k} = \lim_{N\to\infty}\sum_{k=1}^{N}\sqrt{1-\left(\frac{k}{N}\right)}\,\frac{\sqrt{N}}{N}$$

$$= \lim_{N\to\infty}\sum_{k=1}^{N}\sqrt{1-\left(\frac{k}{N}\right)}\cdot\frac{1}{N}\cdot\sqrt{N}$$

$$= \lim_{N\to\infty}\underbrace{\sqrt{N}}_{\downarrow \text{ as } N\to\infty,\ \infty}\cdot\underbrace{\sum_{k=1}^{N}\underbrace{\sqrt{1-\frac{k}{N}}}_{f(x)=\sqrt{1-x}}\cdot\underbrace{\frac{1}{N}}_{\Delta x}}_{\to \text{ area}}$$

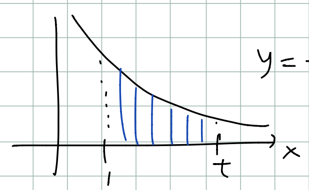




$$= +\infty.$$

$\begin{cases} \cdot \text{ The Definite integral: a warm-up discussion.} \\ \cdot \text{ Uniform continuity} \end{cases}$ part of § 5.2 ~ 3. Appendix IV.

Recall previous example from Lec 2.



$$\text{area} \overset{\text{approx.}}{\approx} \sum_{k=1}^{N} \frac{1}{1+\frac{k(t-1)}{N}} \cdot \frac{t-1}{N}$$

the error $\to 0$ as $N \to \infty$.

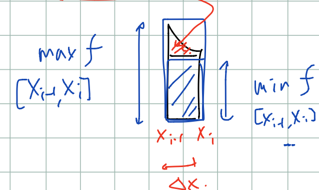
Q Why the error $\to 0$ as $N \to \infty$?

Rigorous Reason Let E_N = the error = $\left| \text{the area} - \sum_{k=1}^{N} \frac{1}{1+k(\frac{t-1}{N})} \frac{t-1}{N} \right|$.

We want to show "For each $\varepsilon > 0$, there exists $N_0 > 0$ such that $\forall N \geq N_0$, $E_N \leq \varepsilon$."

observe E_N = sum of area( ) from each vertical strip.

$$\leq \sum_{i=1}^{N} \left(\max_{[x_{i-1},x_i]} f - \min_{[x_{i-1},x_i]} f \right) \Delta x_i$$



Here, to show this error is small, we can use continuity of f.

First, Note by continuity of f on $[a,b] = [1,t]$, we have (closed interval!)

(*) "uniform continuity": For each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in [a,b]$ with $|x-y| \leq \delta$.

We will explain this later.

Fix $\varepsilon>0$. Choose $\delta>0$ from $(*)$.

Choose large N_0 so that $\dfrac{t-1}{N_0} \leq \delta$

Then $\forall N \geq N_0$

$$E_N \leq \sum_{i=1}^{N} \left(\max_{[x_{i-1},x_i]} f - \min_{[x_{i-1},x_i]} f \right) \Delta x_i$$

$$\leq \sum_{i=1}^{N} \varepsilon \cdot \Delta x_i$$

$$= \varepsilon \sum_{i=1}^{N} \Delta x_i$$

$$= \varepsilon\,(t-1)$$

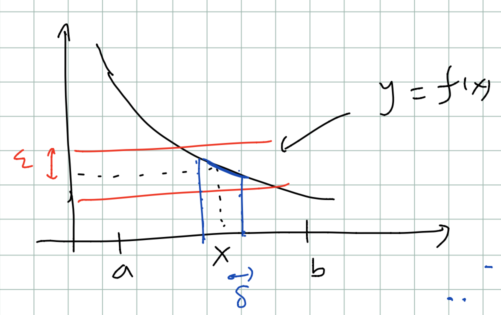
This means rigorously the error $\to 0$ as $N \to \infty$. ∎

The above property $(*)$ is called "uniform continuity"

- Continuity of f on $[a,b]$

$\iff \forall x \in [a,b]$, $\forall \varepsilon>0$, there is $\delta>0$ such that $|f(x)-f(y)| < \varepsilon$ whenever $|x-y|<\delta$.

Here δ may depend on x



- Uniform continuity means that such choice of δ can be made the same for all $x \in [a,b]$.

This holds if f is continuous on $[a,b]$ ← ~~closed interval~~

Thm If $f(x)$ is continuous on $[a,b]$ ← closed interval

then it is uniform continuous on $[a,b]$.

pf See Adams. Appendix IV. (A-29.)

Thm 4 & proof

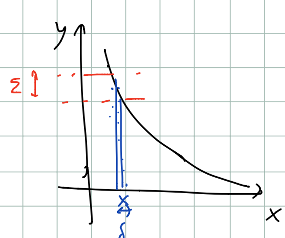
We will skip this proof in class, and will come back if time permits later. $\boxtimes$

Warning In the above thm, closed interval $[a,b]$ is essential.

e.g. $f(x) = \frac{1}{x}$ is continuous on $(0, 1]$

↗ not closed.

It is not uniformly continuous on $(0,1]$.



For $\varepsilon > 0$, need smaller & smaller δ as the pt x gets closer to 0.

Rigorous proof for: $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1]$.

Fix $\varepsilon > 0$, say $\varepsilon = 1$.

Consider $x_n = \frac{1}{n}$. Then $f(x_n) = \frac{1}{x_n} = \frac{1}{1/n} = n$

So to have $|f(y) - f(x)| < 1$, $n-1 < f(y) = \frac{1}{y} < n+1$

Therefore, $\frac{1}{n+1} < y < \frac{1}{n-1}$

That is, $|x_n - y|$ has to be $\leq \max\left(\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n-1} - \frac{1}{n}\right)$.

but RHS $\longrightarrow 0$ as $n \to \infty$.

Thus, we cannot choose one fixed $\delta > 0$, to work for all x_n's :

For any fixed $\delta > 0$, choosing large n with $\max\left(\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n-1} - \frac{1}{n}\right) < \frac{\delta}{2}$ will violate $|x_n - y| < \delta \Rightarrow |f(y) - f(x_n)| < \varepsilon$.

Thus $f(x) = \frac{1}{x}$ is not unif. continuous on $(0, 1]$. $\blacksquare$