

SURFACES WITH BOUNDARY: THEIR UNIFORMIZATIONS, DETERMINANTS OF LAPLACIANS, AND ISOSPECTRALITY

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ABSTRACT. Let Σ be a compact surface of type (g, n) , $n > 0$, obtained by removing n disjoint disks from a closed surface of genus g . Assuming $\chi(\Sigma) < 0$, we show that on Σ , the set of flat metrics which have the same Laplacian spectrum of Dirichlet boundary condition is compact in the C^∞ topology. This isospectral compactness extends the result of Osgood, Phillips, and Sarnak [O-P-S3] for type $(0, n)$ surfaces, whose examples include bounded plane domains.

Our main ingredients are as following. We first show that the determinant of the Laplacian is a proper function on the moduli space of geodesically bordered hyperbolic metrics on Σ . Secondly, we show that the space of such metrics is homeomorphic (in the C^∞ -topology) to the space of flat metrics (on Σ) with constantly curved boundary. Because of this, we next reduce the complicated degenerations of flat metrics to the simpler and well-known degenerations of hyperbolic metrics, and we show that determinants of Laplacians of flat metrics on Σ , with fixed area and boundary of constant geodesic curvature, give a proper function on the corresponding moduli space. This is interesting because Khuri [Kh] showed that if the boundary length (instead of the area) is fixed, the determinant is not a proper function when Σ is of type (g, n) , $g > 0$; while Osgood, Phillips, and Sarnak [O-P-S3] showed the properness when $g = 0$.

1. INTRODUCTION

Kac's [Ka] famous question, 'Can one hear the shape of a drum?' asks whether we can determine a Riemannian manifold by knowing its Laplacian spectrum. Although some Riemannian manifolds are determined uniquely by their spectra (for example, the two dimensional round sphere), there are many counter-examples, in particular, continuous families of Riemannian metrics on some compact manifolds which are isospectral but not (locally) isometric. (See Gordon's survey article [Go].) Thus it is important to know the size of the set of all the Riemannian metrics with the same spectrum on a given compact manifold. This paper addresses the question of whether this isospectral set is compact in the C^∞ -topology. A sequence $\{\sigma_i\}$ of Riemannian metrics on a compact manifold M is said to converge to a Riemannian metric σ on M in the C^∞ -topology if there exist diffeomorphisms F_i of M such that $F_i^* \sigma_i$ converge to σ in the C^∞ sense. In particular, metrics in a compact set in the C^∞ -topology are all quasi-isometric by uniform constants.

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This paper focuses on Riemannian metrics on compact orientable bordered surfaces and their Laplacians (denoted Δ) on functions with Dirichlet boundary condition. The following is the first main conclusion.

Theorem 1.1. *(See Theorem 10.1.) Let Σ be a compact orientable surface with boundary and assume the Euler characteristic $\chi(\Sigma) < 0$. The set of all the smooth flat (i.e. zero Gaussian curvature) metrics on Σ which have the same Laplacian spectrum of Dirichlet boundary condition is compact in the C^∞ -topology.*

Theorem 1.1 extends one of the famous results of Osgood, Phillips, and Sarnak, who showed C^∞ -compactness for isospectral sets of bounded plane domains [O-P-S3]. (They also showed C^∞ -compactness of isospectral sets [O-P-S2] for closed surfaces.) Theorem 1.1 allows us to deal with new examples such as the flat surfaces obtained by removing arbitrary (smooth) neighborhoods of each vertices from compact 2-dimensional simplicial complexes which are manifolds. The topologies of these examples can be much more complicated than those of plane domains.

As in the works of Osgood, Phillips, and Sarnak, the *determinant* $\det \Delta$ of the Laplacian is used as our main analytical notion. It was first introduced by Ray and Singer [R-S1][R-S2] and has become one of the central objects in geometric analysis, algebraic geometry, and string theory. It is defined (see Section 2.2) by using the analytic continuation of the spectral zeta function

$$\zeta(s) = \sum_{0 < \lambda \in \text{Spec}(\Delta)} \lambda^{-s}$$

to the origin and the formula

$$-\log \det \Delta = \zeta'(0).$$

Osgood, Phillips, and Sarnak call $-\log \det \Delta$ the *height* of the Riemannian metric. They use the height as a function on the moduli space of Riemannian metrics to study isospectral problems, given the obvious fact that isospectral metrics have the same height.

Osgood, Phillips, and Sarnak first analyzed the extremal properties of the height function h in each conformal class of a surface and showed the uniformization theorem [O-P-S1]. Namely, in each conformal class of Riemannian metrics on a compact surface, if there is no boundary, there is a unique metric of constant curvature; if the surface has boundary, there is a unique *uniform metric of type I*, i.e. a constant curvature metric with geodesic boundary, and a unique *uniform metric of type II*, i.e. a flat metric with constant geodesic curvature boundary. These uniform metrics realize the minimum of the height under certain constraints.

The above uniformization theorem allows Osgood, Phillips, and Sarnak to reduce the isospectral compactness problem to the properness of the height function on the moduli space of uniform metrics [O-P-S2] [O-P-S3]. This properness was proved by Wolpert [Wo] (also by Bismut and Bost [B-B] in algebraic geometry context) for closed hyperbolic surfaces, and by Osgood, Phillips, and Sarnak on the moduli spaces of uniform metrics of type II with fixed boundary length on punctured spheres [O-P-S3]. The proof of Osgood et al. is quite involved, mainly due to the

complicated degeneration patterns of flat metrics, in comparison to cases involving hyperbolic surfaces (or uniform metrics of type I for nonempty boundary case) where degenerations occur simply when one pinches closed geodesics (*thick-thin decomposition*). Moreover, this properness due to Osgood et al. cannot be extended to the higher genus case as Khuri [Kh] showed that the height is *not* a proper function on the moduli space of uniform metrics of type II with fixed boundary length when the base surface is of type (g, n) , $g > 0$, contrasting to the case of type $(0, n)$ surfaces in [O-P-S3]. A surface of type (g, n) is the surface obtained by removing n disjoint disks from a closed surface of genus g .

Our idea is to use the analysis of hyperbolic side (uniform metrics of type I) to get results for the flat side (uniform metrics of type II). To describe the key lemma for this connection, first denote $\mathbf{M}_I(\Sigma, A)$ and $\mathbf{M}_{II}(\Sigma, A)$ (see Definition 8.2) as the space of uniform metrics of type I and type II, respectively, with fixed area A on a compact orientable surface Σ with boundary. These spaces induce the corresponding moduli spaces $\mathcal{M}_I(\Sigma, A)$, $\mathcal{M}_{II}(\Sigma, A)$, after taking quotient by the group $\text{Diff}(\Sigma)$ of diffeomorphisms of Σ . Then the following theorem is proved.

Theorem 1.2. (See Theorem 8.1) *Let Σ be a compact orientable surface with boundary and assume $\chi(\Sigma) < 0$. The two spaces $\mathbf{M}_I(\Sigma, A)$ and $\mathbf{M}_{II}(\Sigma, A)$ are homeomorphic in the C^∞ -topology, and so are $\mathcal{M}_I(\Sigma, A)$ and $\mathcal{M}_{II}(\Sigma, A)$.*

An important fact used in the hyperbolic (or type I) case is the following theorem concerning the properness of the height of hyperbolic surfaces with geodesic boundary.

Theorem 1.3. (See Corollary 3.4 or Theorem 8.3) *Let \mathcal{M} be the moduli space of compact hyperbolic (Gaussian curvature $\equiv -1$) surfaces with geodesic boundary. The height function h on \mathcal{M} is proper, i.e.*

$$h(M) \rightarrow +\infty$$

as the isometry class $[M]$ approaches $\partial\mathcal{M}$.

This theorem is a corollary of an asymptotic inequality for the height (see Theorem 3.3) which we obtain using the so called *insertion lemma*. This lemma, first introduced by Sarnak [Sa1], uses thick-thin decompositions of hyperbolic surfaces. Our asymptotic inequality partially extends the asymptotic formula of Wolpert [Wo] or of Bismut and Bost [B-B] (see also [Lu]).

To prove Theorem 1.1 we use Theorem 1.2, Theorem 1.3, and a method of Osgood, Phillips, and Sarnak [O-P-S3], which uses the work by Melrose [Me] for boundary geodesic curvature of isospectral flat surfaces. Our approach gives both an extension of the result in [O-P-S3] and a simpler treatment.

On the height of uniform metrics of type II (or flat metrics with boundary of constant geodesic curvature), we easily get the following theorem from Theorem 1.2 and Theorem 1.3.

Theorem 1.4. (See Theorem 11.1.) Suppose $\chi(\Sigma) < 0$. For each $A > 0$, the height h is a proper function on the moduli space $\mathcal{M}_{II}(\Sigma, A)$, i.e.

$$h(M) \rightarrow +\infty$$

as the isometry class $[M]$ approaches $\partial\mathcal{M}_{II}(\Sigma, A)$.

This result is remarkable in comparison with the above non-properness result of Khuri [Kh]. It should be interesting to see the reason why the two conditions, one fixes the boundary length and the other fixes the area, result so differently in the heights.

Notice that the properness in Theorem 1.3 and Theorem 1.4 have another interesting feature: they insure the existence of the global minimum. On the variational study of the height, Sarnak conjectures that it is a Morse function on the Teichmüller space (see [Sa2]). There is also a recent work by Sarnak and Strömbergsson regarding critical points of the height function on the space of (n -dimensional) flat tori [S-St].

This paper relies heavily on the methods and techniques developed by Osgood, Phillips, and Sarnak [O-P-S1] [O-P-S2] [O-P-S3] (see also the good survey paper [Sa1] by Sarnak); however, only the relatively easy part of their analysis of Polyakov-Alvarez formula (see (7.3)) about the conformal effect of the metric to its height is used.

Plan of the paper. In Section 2, we recall some basic results about heat kernels and heights. Sections 3, 4, 5, and 6 prove Theorem 1.3. Sections 7, 8, and 9 explain and prove Theorem 1.2. Section 10 proves Theorem 1.1. Section 11 shows Theorem 1.4. Finally, we make some further remarks in Section 12.

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2. HEAT KERNELS AND HEIGHTS

In this section let M be a compact Riemannian manifold (possibly $\partial M \neq \emptyset$). For a given Riemannian metric σ on M , the Laplacian is the following second order elliptic operator on functions (with Dirichlet boundary condition when $\partial M \neq \emptyset$): in local coordinates,

$$\Delta = \Delta_\sigma = -\frac{1}{\sqrt{\det(\sigma_{ij})}} \partial_i \sigma^{ij} \sqrt{\det(\sigma_{ij})} \partial_j,$$

where σ^{ij} is the inverse matrix of σ_{ij} . We use ∂_n^σ , or just ∂_n , to denote the outer normal derivative on the boundary. The key result of this section is (2.1), which is used to show the insertion lemma (Proposition 3.2).

2.1. Heat kernels. Let $P = P(x, y, t)$, $t > 0$ be the heat kernel (Dirichlet heat kernel when $\partial M \neq \emptyset$), i.e. the fundamental solution of the heat equation

$$\begin{aligned} \partial_t P(x, y, t) + \Delta_x P(x, y, t) &= 0, \\ \lim_{t \rightarrow 0^+} P(x, y, t) &= \delta_x(y) \\ (P(x, y, t) = 0 \text{ for } x \in \partial M). \end{aligned}$$

We use the convention that if x, y belong to two different connected components, then $P(x, y) = 0$.

One of the fundamental results of heat kernels is the estimate provided by Cheeger, Gromov, and Taylor [C-G-T] (see also [Ch]).

Theorem 2.1. *For a complete Riemannian manifold M^n , and $x, y \in M$, $r > 0$ such that the geodesic distance $d(x, y) > 2r$, the following inequality holds.*

$$P(x, y, t) \leq c(n)(t^{-n/2} + tr^{-(n+2)})(\Phi(x, r)\Phi(y, r))^{-1/2} \exp\left(-\frac{(d(x, y) - 2r)^2}{4t}\right).$$

Here $\Phi(x, r)$ is the isoperimetric constant of the geodesic ball $B(x, r)$, i.e.

$$\Phi(x, r) = \inf_{\Omega} \frac{\text{vol}(\partial\Omega)^n}{\text{vol}(\Omega)^{n-1}},$$

where Ω ranges over all open submanifolds which have compact closures with smooth boundary in $B(x, r)$.

Remark 2.1. Judge [Ju] applies this heat kernel estimate to show the convergence of heat kernels when the metric of a manifold degenerates. Ji [Ji2] uses a sharper estimate of Li and Yau [L-Y] under an additional assumption that the Ricci curvature is bounded from below. Li and Yau's estimate fits into our situation; however, the weaker but more general estimate of Cheeger, Gromov, and Taylor is enough for our purpose.

Remark 2.2. By a simple argument which resembles doubling, a similar estimate as given in Theorem 2.1 holds for the Dirichlet heat kernel of a compact Riemannian manifold with boundary. Consider the following example. Let the closed cylinder $[0, 1] \times S^1$ have the metric $du^2 + h(u, v)dv^2$, where u is the parameter of $[0, 1]$, v is the parameter of S^1 , and the function $h(u, v)$ is smooth and positive on the cylinder. First extend the cylinder to a larger cylinder $[-1/2, 3/2] \times S^1$ and the metric to $du^2 + H(u, v)dy^2$, where the function H extends h smoothly (or in such a way it has as much regularity as we need) such that H is constant near the new boundary $\{-1/2\} \times S^1$ and $\{3/2\} \times S^1$. Then double the larger cylinder $[-1/2, 3/2] \times S^1$ to get a torus, say T . Note that the doubled metric on this torus is smooth. Consider the heat kernel P^T of this torus T with the newly constructed metric. Apply the heat kernel bound of Theorem 2.1 to this heat kernel P^T . Now the original cylinder $[0, 1] \times S^1$ is embedded in the torus T , and the Dirichlet heat kernel P^D of the

original cylinder $[0, 1] \times S^1$ is bounded by the heat kernel P^T of the torus by the maximum principle. Then we get

$$P^D(x, y, t) \leq P^T(x, y, t) \leq \text{const. exp}(-\text{const.}/t) \quad (0 < t < 1)$$

where by Theorem 2.1 the constants (> 0) depend only on the distance between x and y and on some appropriate metric balls about these two points. The same inequality

$$(2.1) \quad P(x, y, t) \leq \text{const. exp}(-\text{const.}/t) \quad (0 < t < 1)$$

with the same dependency of the constants as above, holds for a compact Riemannian manifold with piecewise smooth and Lipschitz boundary; in particular, it holds for a compact hyperbolic surface with piecewise geodesic boundary.

Theorem 2.1 and Remark 2.2 are used later to give a proof of the so-called insertion lemma (Proposition 3.2).

2.2. Heights. For a smooth compact Riemannian manifold M with $\partial M \neq \emptyset$, define the spectral zeta function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \int_M P(x, x, t) dx dt, \quad \text{Re } s > \dim M/2,$$

where dx is the Riemannian volume form. The integral $\int_M P(x, x, t) dx$ is called the *trace* of (Dirichlet) heat kernel, and we use $\text{Tr}_D e^{-t\Delta}$ to denote it, where the subscript D means the Dirichlet condition. The spectral zeta function $\zeta(s)$ has its meromorphic extension to \mathbb{C} and is holomorphic at $s = 0$. In two-dimension, its regularity at $s = 0$ can easily be seen by using the asymptotic formula by McKean and Singer (see [O-P-S1] section 1, [M-S] (5.2))

$$(2.2) \quad \int_M P(x, x, t) dx = \frac{1}{4\pi t} \text{Area}(M) - \frac{1}{8\sqrt{\pi t}} \int_{\partial M} ds + \frac{1}{12\pi} \int_M K dx - \frac{1}{12\pi} \int_{\partial M} k ds + o(\sqrt{t}) \quad (\text{as } t \rightarrow 0),$$

where K, k denote the Gaussian curvature and the boundary geodesic curvature, respectively, and ds denotes the arc length element.

Define the determinant $\det \Delta$ of the Laplacian as following:

$$-\log \det \Delta = \zeta'(0).$$

We call $-\log \det \Delta$ the *height* of M and denote it $h(M)$.

Remark 2.3. It is not hard to see that the height h gives a continuous function on the space of Riemannian metrics in the C^∞ -topology. This is because for a given time $t > 0$ the heat kernel $P(x, y, t)$ depends continuously on the metric and

$$\begin{aligned} P(x, y, t) &= \int_M P(x, z, t/2) P(z, y, t/2) dz \\ &\leq e^{-\lambda_1(t-T)} \sqrt{P(x, x, T)} \sqrt{P(y, y, T)}, \quad \text{for } t \geq T > 0, \end{aligned}$$

and $P(x, x, t) = O(t^{-\dim M/2})$ as $t \rightarrow 0$, where the first eigenvalue λ_1 and the constant of $O(t^{-\dim M/2})$ depend continuously on the metric. Therefore, $\zeta(s)$ and

$\frac{d}{ds}\zeta(s)$ for each $\operatorname{Re} s > \frac{\dim M}{2}$ are continuous with respect to the metric, and their analytic extensions at $s = 0$ are continuous on the metric as well.

3. HEIGHTS OF BORDERED HYPERBOLIC SURFACES

Theorem 1.3 is proven in this section. Let M be a hyperbolic surface (Gaussian curvature -1) with geodesic boundary, i.e. $\partial M = \cup_{i=1}^n b_i$, where each b_i is a closed curve with zero geodesic curvature. Note that by Gauss-Bonnet theorem, $\chi(M) < 0$. As typical in analysis of hyperbolic surfaces, the so-called *thick and thin decomposition* is used. The insertion lemma (Proposition 3.2) is applied to this decomposition. The method of using the insertion lemma to study the height of hyperbolic surfaces is first considered by Sarnak [Sa1].

Let \tilde{M} be the double of M , then \tilde{M} is a smooth closed hyperbolic surface. Let g be the genus of \tilde{M} . It is a well-known fact (see, for example, [Wo]) that there is a constant $1 > c_* > 0$, depending only on g , with the following property. There are only finitely many (at most $3g - 3$) closed primitive geodesics of \tilde{M} , say, $\gamma_1, \dots, \gamma_k$, whose lengths $l(\gamma_i)$ are less than c_* (these geodesics are called *short geodesics*); for each γ_i , there is a tubular neighborhood C_i called *standard collar* [Wo], of width

$$\sinh^{-1}(1/\sinh(\frac{1}{2}l(\gamma_i))) \approx 2 \log \frac{2}{l(\gamma_i)};$$

each C_i is a hyperbolic cylinder with the core geodesic γ_i , and these collars are all mutually disjoint. The standard collar C_γ of a short geodesic γ of length l can be regarded as a domain $[0, l] \times [l, \pi - l]$ with variables (u, v) such that $u = 0$ is identified with $u = l$. The hyperbolic metric is $1/\sin^2(v)(du^2 + dv^2)$. The *standard subcollar* SC_γ is defined as the subset

$$\{(u, v) \in C_\gamma \mid 2l \leq v \leq \pi - 2l\} / (u = 0 \sim u = l) \subset C_\gamma.$$

From the argument given by Wolpert (see [Wo], section 2.6 and 2.7, especially p296), the surface $\tilde{M} \setminus \bigcup_{\gamma_i} SC_{\gamma_i}$ has uniformly bounded geometry, i.e. the set $\{\tilde{M} \setminus \bigcup_{\gamma_i} SC_{\gamma_i}\}$ of such surfaces forms a compact set in the C^∞ topology of the space of Riemannian manifolds modulo isometries. Moreover, there exists a constant $\delta^* = \delta^*(c^*) > 0$ such that the tubular neighborhood

$$A_\gamma = \{x \in C_\gamma \mid \operatorname{dist}(x, \partial SC_\gamma) \leq \delta^*\} \subset C_\gamma$$

has uniformly bounded geometry.

For a short geodesic γ in the double \tilde{M} , there are only three possible cases:

- I : $\gamma \cap \partial M = \emptyset$,
- II : $\gamma \subset \partial M$,
- III : $\gamma \pitchfork \partial M$.

It is easy to see that in the second case, $SC_\gamma \setminus \partial M$ consists of two isometric hyperbolic cylinders, whose ends are γ and a part of ∂SC . In the third case, $SC_\gamma \setminus \partial M$ consists of two isometric hyperbolic 4-gons, each contains half of γ .

Proposition 3.1. *Let SC_I , SC_{II} , SC_{III} denote the regions $M \cap (SC_\gamma \setminus \partial M)$ corresponding to geodesics γ in cases I, II, III, respectively. Their heights satisfy:*

$$(3.1) \quad h(SC_I) \sim \frac{\pi^2}{6l(\gamma)} + \log l(\gamma) + O(1) \quad \text{by Lundelius [Lu] and Sarnak [Sa1],}$$

$$(3.2) \quad h(SC_{II}) \sim \frac{\pi^2}{12l(\gamma)} + \log l(\gamma) + O(1),$$

$$(3.3) \quad h(SC_{III}) \sim \frac{\pi^2}{12l(\gamma)} + O(1).$$

Proof. The proof is given in Section 5. □

Remark 3.1. The domain SC_{III} has right-angle corners; thus the previous definition of height in Section 2.2 does not directly apply. This subtlety will be addressed in Section 4.

Proposition 3.2. (Insertion Lemma) *Let N and A be the subsets $M \cap (\bigcup_{\gamma_i} SC_{\gamma_i})$ and $M \cap (\bigcup_{\gamma_i} A_{\gamma_i})$ of \tilde{M} , respectively. Then*

$$(3.4) \quad \text{the heights } h(N), h(M \setminus N), \text{ and } h(A) \text{ are defined;}$$

$$(3.5) \quad h(M) \geq h(N) + h(M \setminus N) + O(1),$$

where the constant depends only on A and $\overline{\partial N \setminus \partial M}$.

Proof. This result is a modified version of Osgood-Phillips-Sarnak's insertion lemma which was proved for the flat surface case [O-P-S3] (see also [Kh]). The proof is given in Section 6. □

$M \setminus N$ and A have uniformly bounded geometry and so their heights are also bounded uniformly. As a consequence of Proposition 3.2 and Proposition 3.1, we get the following asymptotic inequality.

Theorem 3.3. *With the above setting,*

$$\begin{aligned} h(M) \geq & \sum_{\gamma_i \cap \partial M = \emptyset} \left(\frac{\pi^2}{6l(\gamma_i)} + \log l(\gamma_i) \right) \\ & + \sum_{\gamma_i \subset \partial M} \left(\frac{\pi^2}{12l(\gamma_i)} + \log l(\gamma_i) \right) + \sum_{\gamma_i \pitchfork \partial M} \frac{\pi^2}{12l(\gamma_i)} + O(1). \end{aligned}$$

Remark 3.2. An explicit formula of the height for geodesically bordered hyperbolic surfaces is given in terms of Selberg's zeta function by Bolte and Steiner [B-S]; however, it seems quite complicated to use their formula to get an asymptotic estimation. In the case of closed hyperbolic surfaces, Wolpert [Wo] succeeded in using Selberg's zeta function expression of the height; his proof was later simplified by Lundelius [Lu], whose method is in the same spirit as ours in using the idea of insertion lemma. Bismut and Bost [B-B] took an alternative algebraic geometry approach.

Remark 3.3. One may try to refine the estimate in Theorem 3.3 by adding contributions from the low eigenvalues as in [Lu] [Wo].

Corollary 3.4. (Theorem 1.3) *On the moduli space \mathcal{M} of compact hyperbolic (Gaussian curvature $\equiv -1$) surfaces with geodesic boundary, the function h is proper, i.e.*

$$h(M) \rightarrow +\infty$$

as the isometry class $[M]$ approaches $\partial\mathcal{M}$.

For a (finite dimensional) topological space \mathcal{T} , we say that a sequence $t_i \rightarrow \partial\mathcal{T}$ if the set $\{t_i\}_{i=0}^\infty$ is not contained in any compact subset of \mathcal{T} .

4. SEPARATION OF VARIABLES, HEAT KERNELS, AND HEIGHTS

Domains like SC_{III} have points of special type singularity in their boundaries: there are four right-angle corners in ∂SC_{III} . In this section we study the heat kernels and heights of such domains. The subtlety due to their singularity can be resolved by the separation of variables technique. (See [Ji1] for a different but more extensive use of separation of variables in studying the spectrum of a Riemann surface.)

4.1. 1-dimensional heat traces. Before we proceed the separation of variables, let's consider the 1-dimensional case.

Let $[A, B] \subset \mathbb{R}$ be a finite closed interval with metric dx . For a smooth function ϕ on $[A, B]$, the Laplace operator of the metric $e^\phi dx$ is given by

$$\square_\phi = -e^{-2\phi} \left[\frac{d}{dx} \right]^2 + \phi' e^{-2\phi} \frac{d}{dx}.$$

Let $Q_\phi = -e^{-2\phi} \left[\frac{d}{dx} \right]^2$, then

$$(4.1) \quad Q_\phi = \square_\phi - \phi' e^{-2\phi} \frac{d}{dx}.$$

Let $e(x, y, t)$ be the fundamental solution of the heat equation of Q_ϕ :

$$\begin{aligned} \frac{\partial}{\partial t} e(x, y, t) + [Q_\phi]_x e(x, y, t) &= 0, \\ \lim_{t \rightarrow 0} e(x, y, t) &= \delta_x(y), \\ e(x, y, t) &= 0 \text{ for } x \in \{A, B\}, \end{aligned}$$

where δ_x is the Dirac δ -function with respect to the metric $e^\phi dx$, i.e. $\int \delta_x(y) f(y) e^{\phi(y)} dy = f(x)$, for every smooth function f .

Let φ be an arbitrary smooth function on $[A, B]$. Define

$$(4.2) \quad \text{Tr}_D \varphi e^{-tQ_\phi} = \int_A^B \varphi(x) e(x, x, t) e^{\phi(x)} dx,$$

where the subscript D means the Dirichlet boundary condition. By using a result of McKean and Singer ([M-S], pp 53. equation (5.2) and its proof, especially pp 55–56) applied to (4.1), we have

$$(4.3) \quad \text{Tr}_D \varphi e^{-tQ_\phi} = \frac{1}{\sqrt{4\pi t}} \int_A^B \varphi(x) e^{\phi(x)} dx - \frac{1}{4} (\varphi(A) + \varphi(B)) + O(\sqrt{t}) \quad (\text{as } t \rightarrow 0).$$

Define the zeta function of Q_ϕ :

$$Z_\phi(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_D e^{-tQ_\phi} dt.$$

By (4.3), the regularization of $Z_\phi(s)$ at $s = 0$ is enabled. Now, by an argument identical to the derivation of the Polyakov formula (see for example, [O-P-S1], pp 155–156),

$$(4.4) \quad Z'_\phi(0) = -\frac{1}{2}(\phi(A) + \phi(B)) + Z'_0(0).$$

4.2. Separation of variables and heat kernels. Let's first fix some notation.

Definition 4.1. For $0 < l < \frac{\pi}{4}$ and $l \leq A < B \leq \pi - l$, let $C^{l,[A,B]}$ be the smooth hyperbolic cylinder defined as the domain $[0, l] \times [A, B]$ with variables (u, v) , $u = 0$ identified with $u = l$, and with the hyperbolic metric $1/\sin^2(v)(du^2 + dv^2)$. For example, $SC_I = C^{l,[2l,\pi-2l]}$, $SC_{II} = C^{l,[2l,\pi/2]}$. Let $C_{III}^{l,[A,B]}$ be the subset

$$C_{III}^{l,[A,B]} = \{(u, v) \in C^{l,[A,B]} \mid 0 \leq u \leq l/2\} \subset C^{l,[A,B]}.$$

For example, $SC_{III} = C_{III}^{l,[2l,\pi-2l]}$.

We have the L^2 -decomposition

$$L^2(C^{l,[A,B]}) = L^2([A, B], \frac{1}{\sin^2(v)} dv) \oplus \left[\bigoplus_{m \in \mathbb{N}} L_{m,1}^2(C^{l,[A,B]}) \oplus L_{m,2}^2(C^{l,[A,B]}) \right]$$

where

$$L_{m,1}^2(C^{l,[A,B]}) = \{f(v) \sqrt{\frac{2}{l}} \cos(2\pi m u/l) \in L^2(C^{l,[A,B]})\} \simeq L^2([A, B], \frac{1}{\sin^2(v)} dv),$$

$$L_{m,2}^2(C^{l,[A,B]}) = \{f(v) \sqrt{\frac{2}{l}} \sin(2\pi m u/l) \in L^2(C^{l,[A,B]})\} \simeq L^2([A, B], \frac{1}{\sin^2(v)} dv).$$

The Laplace operator

$$\Delta = -\sin^2(v) \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right)$$

has the decomposition

$$(\Delta, L^2(C^{l,[A,B]})) \simeq \bigoplus_{m \in \mathbb{Z}} (\Delta_l(m), L^2([A, B], \frac{1}{\sin^2(v)} dv)),$$

where

$$\Delta_l(m) = -\sin^2(v) \left(\frac{\partial^2}{\partial v^2} - \frac{4\pi^2 m^2}{l^2} \right).$$

Denote the Dirichlet boundary condition with subscript D , then

$$(\Delta, L_D^2(C^{l,[A,B]})) \simeq \bigoplus_{m \in \mathbb{Z}} (\Delta_l(m), L_D^2([A, B], \frac{1}{\sin^2(v)} dv)).$$

Regarding $C_{III}^{l,[A,B]}$, it is easy to see that

$$\begin{aligned} L_D^2(C_{III}^{l,[A,B]}) &\simeq \bigoplus_{m \in \mathbb{N}} (L_{m,2,D}^2(C^{l,[A,B]}), \frac{1}{\sin^2(v)} dv), \\ (\Delta, L_D^2(C_{III}^{l,[A,B]})) &\simeq \bigoplus_{m \in \mathbb{N}} (\Delta_l(m), L_D^2([A,B], \frac{1}{\sin^2(v)} dv)). \end{aligned}$$

Let $P^{l,[A,B]}(u_1, v_1; u_2, v_2; t)$, $P_{III}^{l,[A,B]}(u_1, v_1; u_2, v_2; t)$, and $P^{l,m}(v_1, v_2; t)$ denote the Dirichlet heat kernels of the operators $(\Delta, L_D^2(C^{l,[A,B]}))$, $(\Delta, L_D^2(C_{III}^{l,[A,B]}))$, and $(\Delta_l(m), L_D^2([A,B], \frac{1}{\sin^2(v)} dv))$, respectively. We see that

$$\begin{aligned} &P^{l,[A,B]}(u_1, v_1; u_2, v_2; t) \\ &= P^{l,0}(v_1, v_2; t) \frac{1}{l} \\ &+ \sum_{m \in \mathbb{N}} P^{l,m}(v_1, v_2; t) \frac{2}{l} \left(\cos\left(\frac{2\pi m u_1}{l}\right) \cos\left(\frac{2\pi m u_2}{l}\right) + \sin\left(\frac{2\pi m u_1}{l}\right) \sin\left(\frac{2\pi m u_2}{l}\right) \right) \end{aligned}$$

and

$$P_{III}^{l,[A,B]}(u_1, v_1; u_2, v_2; t) = \sum_{m \in \mathbb{N}} P^{l,m}(v_1, v_2; t) \frac{4}{l} \sin\left(\frac{2\pi m u_1}{l}\right) \sin\left(\frac{2\pi m u_2}{l}\right).$$

4.3. Heat traces, asymptotic expansions, and heights. Let $\varphi(v)$ be a non-negative smooth function on $[A, B]$ which is constant near the boundary points $v = A$ and $v = B$: for example, $\varphi \equiv 1$. Let's consider modified heat traces of the form $\text{Tr} \varphi e^{-t\Delta}$. First,

$$\begin{aligned} &\text{Tr}_{L_D^2(C^{l,[A,B]})} \varphi e^{-t\Delta} \\ &= \int_A^B \int_0^l \varphi(v) P^{l,[A,B]}(u, v; u, v; t) \frac{1}{\sin^2(v)} du dv \\ &= \int_A^B \varphi(v) P^{l,0}(v, v; t) \frac{1}{\sin^2(v)} dv + 2 \int_A^B \varphi(v) \sum_{m \in \mathbb{N}} P^{l,m}(v, v; t) \frac{1}{\sin^2(v)} dv. \end{aligned}$$

Similarly,

$$\text{Tr}_{L_D^2(C_{III}^{l,[A,B]})} \varphi e^{-t\Delta} = \int_A^B \varphi(v) \sum_{m \in \mathbb{N}} P^{l,m}(v, v; t) \frac{1}{\sin^2(v)} dv.$$

Thus, we see

$$(4.5) \quad 2 \text{Tr}_{L_D^2(C_{III}^{l,[A,B]})} \varphi e^{-t\Delta} = \text{Tr}_{L_D^2(C^{l,[A,B]})} \varphi e^{-t\Delta} - \text{Tr}_{L_D^2([A,B])} \varphi e^{-t\Delta_l(0)},$$

where

$$\text{Tr}_{L_D^2([A,B])} \varphi e^{-t\Delta_l(0)} = \int_A^B \varphi(v) P^{l,0}(v, v; t) \frac{dv}{\sin^2(v)}.$$

On the other hand, by the result ([M-S] pp. 53–60, equation (5.2) and its proof) of McKean and Singer, it follows the asymptotic expansion

$$(4.6) \quad \begin{aligned} & \text{Tr}_{L_D^2(C^{l,[A,B]})} \varphi e^{-t\Delta} \\ &= \frac{l}{4\pi t} \int_A^B \varphi(v) \frac{dv}{\sin^2(v)} - \frac{l}{8\sqrt{\pi t}} \left(\frac{\varphi(A)}{\sin A} + \frac{\varphi(B)}{\sin B} \right) \\ &+ \frac{l}{12\pi} \int_A^B \frac{-\varphi(v)}{\sin^2(v)} dv - \frac{l}{12\pi} \left(\frac{\varphi(A)k(A)}{\sin A} + \frac{\varphi(B)k(B)}{\sin B} \right) + o(\sqrt{t}) \quad (\text{as } t \rightarrow 0), \end{aligned}$$

where $k(A)$, $k(B)$ denote the constant geodesic curvatures of the boundaries $v = A$ and $v = B$ of $C^{l,[A,B]}$.

Remark 4.1. The above asymptotic expansion (4.6) (up to $o(\sqrt{t})$) uses that near the boundary the modifying function φ is constant along the normal direction. However, the 1-dimensional expansion (4.3) (up to $O(\sqrt{t})$) needs no such restriction on φ . We expect that for general case the asymptotic expansion would contain some derivatives of φ (with respect to the normal direction to the boundary).

For the asymptotic expansion of $\text{Tr}_{L_D^2([A,B])} \varphi e^{-t\Delta_i(0)}$, apply (4.3) with $\phi = -\log \sin v$ to get

$$(4.7) \quad \text{Tr}_{L_D^2([A,B])} \varphi e^{-t\Delta_i(0)} = \frac{1}{\sqrt{4\pi t}} \int_A^B \frac{\varphi(v)}{\sin v} dv - \frac{1}{4}(\varphi(A) + \varphi(B)) + O(\sqrt{t}) \quad (\text{as } t \rightarrow 0).$$

Remark 4.2. Note that the fundamental solution e in the integral (4.2) is now $e(v, v, t) = P^{l,0}(v, v; t) \frac{1}{\sin v}$.

From (4.5), (4.6), and (4.7), an asymptotic expansion

$$(4.8) \quad \text{Tr}_{L_D^2(C_{III}^{l,[A,B]})} \varphi e^{-t\Delta} = \frac{C_1}{t} + \frac{C_2}{\sqrt{t}} + C_3 + O(\sqrt{t}) \quad (\text{as } t \rightarrow 0)$$

follows; this shows the spectral zeta function

$$Z_{III}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{L_D^2(C_{III}^{l,[A,B]})} e^{-t\Delta} dt$$

has regularization at $s = 0$; thus, the well-definedness of the height $h(C_{III}^{l,[A,B]})$ follows. By (4.5),

$$(4.9) \quad 2h(C_{III}^{l,[A,B]}) = h(C^{l,[A,B]}) - Z'(0).$$

Here,

$$Z(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}_{L_D^2([A,B])} e^{-t\Delta_i(0)} dt,$$

and its regularization at $s = 0$ is enabled by the asymptotic expansion (4.7).

5. PROOF OF PROPOSITION 3.1

(3.1) was shown by Lundelius ([Lu], section 3.3) and Sarnak ([Sa1], Appendix) by using the Polyakov-Alvarez formula (see (7.3)) for a flat cylinder. Exactly the same proof as in [Lu] can show (3.2).

For (3.3), we apply the results of Section 4. Note that for a short geodesic of length l , the domains SC_I , SC_{III} are exactly $C^{l,[2l,\pi-2l]}$, $C_{III}^{l,[2l,\pi-2l]}$, respectively. By (4.9) for $A = 2l, B = \pi - 2l$,

$$(5.1) \quad 2h(SC_{III}) = h(SC_I) - Z'(0).$$

By (4.4) for $\phi = \log(1/\sin(v))$,

$$Z'(0) = Z'_\phi(0) = \log(\sin(2l)) + Z'_0(0)$$

where $Z'_0(0) = 2\log\left(\frac{\pi-4l}{\pi}\right)\zeta_R(0) + 2\zeta'_R(0)$ for the Riemann zeta function $\zeta_R(s)$. Hence,

$$Z'(0) \sim \log 2l + O(1), \quad \text{as } l \rightarrow 0,$$

and so by (5.1) and (3.1) the proof of (3.3) is complete.

6. PROOF OF PROPOSITION 3.2 (INSERTION LEMMA)

The proof consists of two parts. First, we establish that the definition of height h makes sense for N , $M \setminus N$, and A , whose boundaries have corners of special type. Then we show the inequality (3.5).

6.1. Proof of (3.4). Recall that $N = M \cap (\bigcup_{\gamma_i} SC_{\gamma_i})$ and A is the tubular neighborhood $M \cap (\bigcup_{\gamma_i} A_{\gamma_i})$ of $\partial N \setminus \partial M$. So, the connected components of N and A are of the form either $C^{l,[A,B]}$ or $C_{III}^{l,[A,B]}$ (see Definition 4.1). The results of Section 4 (especially (4.9)) thus show that the values $h(N)$ and $h(A)$ are well-defined.

We now show the well-definedness of the value $h(M \setminus N)$. Let V , V_0 , V_1 and $Q(x, y, t)$, $Q_0(x, y, t)$, $Q_1(x, y, t)$ denote the domains $M \setminus N$, M , $(M \setminus N) \cap A$ and the corresponding Dirichlet heat kernels, respectively. It is enough to show $\int_V Q(x, x, t)dx$ has an asymptotic expansion of the form

$$(6.1) \quad \frac{C_1}{t} + \frac{C_2}{\sqrt{t}} + C_3 + O(\sqrt{t}) \quad (\text{as } t \rightarrow 0).$$

Choose a smooth function φ , $0 \leq \varphi \leq 1$ supported on A as follows. On each connected component of A (which is of the form either $C^{l,[A,B]}$ or $C_{III}^{l,[A,B]}$), $\varphi = \varphi(v)$ and $\varphi \equiv 1$ near $\partial N \setminus \partial M$; $\varphi \equiv 0$ near the boundary $v = A$ and $v = B$. Then

$$\begin{aligned} & \int_V Q(x, x, t)dx \\ &= \int_{V_1} \varphi Q_1(x, x, t)dx + \int_{V_1} \varphi(Q(x, x, t) - Q_1(x, x, t))dx \\ & \quad + \int_V (1 - \varphi)(Q(x, x, t) - Q_0(x, x, t))dx + \int_V (1 - \varphi)Q_0(x, x, t)dx \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

To deal with I_2 and I_3 , apply the maximum principle to $Q(x, \cdot, t) - Q_i(x, \cdot, t)$, $i = 0, 1$, and use (2.1) to see that

$$(6.2) \quad \begin{aligned} 0 &\leq (1 - \varphi(x))(Q_0(x, x, t) - Q(x, x, t)) \leq (1 - \varphi(x))c_1 \exp\left(-\frac{c_2}{t}\right), \\ 0 &\leq \varphi(x)(Q(x, x, t) - Q_1(x, x, t)) \leq \varphi(x)c_3 \exp\left(-\frac{c_4}{t}\right), \quad (0 < t < 1). \end{aligned}$$

Thus, $\varphi(x)(Q(x, x, t) - Q_1(x, x, t))$ and $(1 - \varphi(x))(Q_0(x, x, t) - Q(x, x, t))$ decrease rapidly as $t \rightarrow 0$ uniformly for $x \in V$;

$$I_2 = o(t), \quad I_3 = o(t) \quad \text{as } t \rightarrow 0.$$

Let's now consider I_1 and I_4 . Note the connected components of V_1 are of the form either $C^{l,[A,B]}$ or $C_{III}^{l,[A,B]}$. Apply (4.6) and (4.8) to see I_1 has an asymptotic expansion of the form (6.1). On the other hand, I_4 has an asymptotic expansion of the form (6.1) by the same reason as for (4.6) (see Remark 4.1). This completes the proof.

6.2. Proof of (3.5). We follow the outline given in the proof of insertion lemma in [O-P-S3] or in [Kh]. Denote M , N , $M \setminus N$ and the corresponding Dirichlet heat kernels by M_0, M_1, M_2 and $P_0(x, y, t), P_1(x, y, t), P_2(x, y, t)$, respectively. Pick a tubular neighborhood A_1 of $\overline{\partial M_1} \setminus \partial M_0$ with $A_1 \subset\subset A$, such that each connected component of A_1 is of the form either $C^{l,[A,B]}$ or $C_{III}^{l,[A,B]}$ (see Definition 4.1). Choose a compactly supported smooth function φ on A_1 , such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ near $\overline{\partial M_1} \setminus \partial M_0$. Then

$$\begin{aligned} &h(M_0) - h(M_1) - h(M_2) \\ &= \frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \sum_{j=1}^2 \int_0^1 \int_{M_j} t^{s-1} \varphi(P_0(x, x, t) - P_j(x, x, t)) dx dt \\ &\quad + \frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \sum_{j=1}^2 \int_0^1 \int_{M_j} t^{s-1} (1 - \varphi)(P_0(x, x, t) - P_j(x, x, t)) dx dt \\ &\quad + \frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \sum_{j=1}^2 \int_1^\infty \int_{M_j} t^{s-1} \varphi(P_0(x, x, t) - P_j(x, x, t)) dx dt \\ &\quad + \frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \sum_{j=1}^2 \int_1^\infty \int_{M_j} t^{s-1} (1 - \varphi)(P_0(x, x, t) - P_j(x, x, t)) dx dt \\ &= (I) + (II) + (III) + (IV). \end{aligned}$$

Let's consider (I). For the regions A , $A \cap M_1$, and $A \cap (M_2)$, denote the corresponding Dirichlet heat kernels by $P_0^A(x, y, t), P_1^A(x, y, t)$, and $P_2^A(x, y, t)$, respectively. Apply the maximum principle to $P_j(x, \cdot, t) - P_j^A(x, \cdot, t)$ and use (2.1), to see that for $i = 0, 1, 2$,

$$(6.3) \quad 0 \leq \varphi(x)(P_j(x, x, t) - P_j^A(x, x, t)) \leq \varphi(x)c_1(A) \exp\left(-\frac{c_2(A)}{t}\right), \quad (0 < t < 1).$$

Now, write

$$\begin{aligned}
(I) &= \frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \sum_{j=1}^2 \int_0^1 \int_{M_j} t^{s-1} \varphi(x) (P_0^A(x, x, t) - P_j^A(x, x, t)) dx dt \\
&\quad + \frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \sum_{j=1}^2 \int_0^1 \int_{M_j} t^{s-1} \varphi(x) (P_0(x, x, t) - P_0^A(x, x, t)) dx dt \\
&\quad - \frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \sum_{j=1}^2 \int_0^1 \int_{M_j} t^{s-1} \varphi(x) (P_j(x, x, t) - P_j^A(x, x, t)) dx dt.
\end{aligned}$$

By (6.3), the second and third terms have uniformly rapidly decreasing integrands supported inside A ; the first term depends only on A and $\overline{\partial M} \setminus \partial N$. Therefore we see that

$$(I) = O(1)$$

with the constant dependent only on A and $\overline{\partial M} \setminus \partial N$.

It remains to show (II), (III), and (IV) are nonnegative. $(1 - \varphi(x))(P_0(x, x, t) - P_j(x, x, t))$ decreases uniformly for $x \in M_j$ rapidly as $t \rightarrow 0$ (by the above argument as for $P_j - P_j^A$) and $t \rightarrow \infty$, and so does $\varphi(x)(P_0(x, x, t) - P_j(x, x, t))$ as $t \rightarrow \infty$. Thus,

$$\begin{aligned}
(II) &= \sum_{j=1}^2 \int_0^1 \int_{M_j} t^{-1} (1 - \varphi) (P_0(x, x, t) - P_j(x, x, t)) dx dt \geq 0, \\
(III) &= \sum_{j=1}^2 \int_1^\infty \int_{M_j} t^{-1} \varphi (P_0(x, x, t) - P_j(x, x, t)) dx dt \geq 0, \\
(IV) &= \sum_{j=1}^2 \int_1^\infty \int_{M_j} t^{-1} (1 - \varphi) (P_0(x, x, t) - P_j(x, x, t)) dx dt \geq 0,
\end{aligned}$$

with the inequalities by the maximum principle. This completes the proof.

7. CONFORMAL METRICS AND HEIGHTS

Let Σ be a compact surface with boundary, and σ, σ_0 be two metrics on Σ conformal with each other, i.e. $\sigma = e^{2\psi} \sigma_0$. Let K, k (resp. K_0, k_0) be the sectional curvature and the boundary geodesic curvature of the metrics σ (resp. σ_0). Then

$$(7.1) \quad K = e^{-2\psi} (\Delta_0 \psi + K_0),$$

$$(7.2) \quad k = e^{-\psi} (k_0 + \partial_n \psi),$$

where ∂_n is the outer normal derivative for $\partial \Sigma$ with respect to the metric σ_0 . We will use the subscript 0 for the quantities of the metric σ_0 .

7.1. Polyakov-Alvarez formula. Alvarez extended (see [Al] pp 158–159 also see [O-P-S1]) the Polyakov formula to surfaces with boundary. Namely, for the metric σ_0 , the height h satisfies

$$(7.3) \quad \begin{aligned} h(e^{2\psi}\sigma_0) = & \frac{1}{6\pi} \left\{ \frac{1}{2} \int_{\Sigma} |\nabla_0 \psi|^2 dA_0 + \int_{\Sigma} K_0 \psi dA_0 + \int_{\partial\Sigma} k_0 \psi ds_0 \right\} \\ & + \frac{1}{4\pi} \int_{\partial\Sigma} \partial_n \psi ds_0 + h(\sigma_0). \end{aligned}$$

In particular, for the scaling $\lambda^2\sigma$, where $\lambda > 0$,

$$(7.4) \quad h(\lambda^2\sigma) = \frac{\chi(\Sigma)}{3} \log \lambda + h(\sigma).$$

8. UNIFORM METRICS

In this section let $\Sigma = \Sigma_{g,n}$ be a type (g, n) compact orientable surface with boundary, i.e. Σ is obtained by removing n disjoint disks from a closed surface of genus g . Assume $\chi(\Sigma) = 2 - 2g - n < 0$.

8.1. Uniform metrics of types I and II. The following definitions are due to Osgood, Phillips, and Sarnak.

Definition 8.1. [O-P-S1] A metric σ on Σ is called a *uniform metric of type I* if the resulting Riemannian manifold M has constant sectional curvature and ∂M is of zero geodesic curvature. The metric is called a *uniform metric of type II* if the resulting Riemannian manifold M is flat, i.e. the sectional curvature is zero, and ∂M is of constant geodesic curvature, with the same constant for all the boundary components.

Remark 8.1. For a uniform metric of type I, the area determines the sectional curvature of the surface; however, it does not determine, for a uniform metric of type II, the boundary geodesic curvature.

Definition 8.2. Let $\mathbf{M}_I(\Sigma, A)$ (resp. $\mathbf{M}_{II}(\Sigma, A)$) denote the space of uniform metrics of type I (resp. II) on Σ with fixed area A . The topologies on these spaces are induced by the natural C^∞ -topology on the space of sections in the bundle $(T^*\Sigma)^2$.

Definition 8.3. Define the moduli space $\mathcal{M}_I(\Sigma, A)$ (resp. $\mathcal{M}_{II}(\Sigma, A)$) of metrics of type I (resp. II) as

$$\begin{aligned} \mathcal{M}_I(\Sigma, A) &= \mathbf{M}_I(\Sigma, A) / \text{Diff}(\Sigma), \\ \mathcal{M}_{II}(\Sigma, A) &= \mathbf{M}_{II}(\Sigma, A) / \text{Diff}(\Sigma), \end{aligned}$$

where $\text{Diff}(\Sigma)$ is the group of diffeomorphisms of Σ .

Remark 8.2. It is known that both $\mathcal{M}_I(\Sigma, A)$ and $\mathcal{M}_{II}(\Sigma, A)$ have real dimension $6g - 6 + 3n$.

Remark 8.3. As discussed in [O-P-S3], a uniform metric of type II is nothing but a metric obtained from a closed flat surface with conical singularities, by removing the metric disks of a fixed radius centered at each conical points. It is known ([O-P-S3],

Theorem 1.1) that the moduli space $\mathcal{M}_{II}(\Sigma, A)$ is in one-to-one correspondence with the subset $\mathcal{C}(\Sigma)'$ of the space $\mathcal{C}(\Sigma)$ of conical metrics—flat metrics with conical singularities—on Σ , such that for each metric in $\mathcal{C}(\Sigma)'$, the distances between conical points are > 2 . We refer the reader to [O-P-S3] or [Kh] for details of conical metrics.

Let σ be a uniform metric of type II with area A on Σ . By the uniformization theorem, we find a unique uniform metric τ of type I with area A within the conformal class of σ (see [O-P-S1] [Ab]):

$$\sigma = e^{2\psi} \tau$$

where ψ uniquely solves the normalized boundary value problem [O-P-S1]

$$\begin{cases} \Delta_\tau \psi + \frac{2\pi\chi(\Sigma)}{A} = 0, & \text{in } \Sigma, \\ \partial_n^\tau \psi - \frac{2\pi\chi(\Sigma)e^\psi}{\int_{\partial\Sigma} e^\psi ds_\tau} = 0, & \text{on } \partial\Sigma, \\ \int_\Sigma e^{2\psi} dA_\tau = A. \end{cases}$$

This ψ also solves uniquely

$$\begin{cases} \Delta_\sigma \psi + \frac{2\pi\chi(M)e^{-2\psi}}{A} = 0, & \text{in } \Sigma, \\ \partial_n^\sigma \psi - \frac{2\pi\chi(\Sigma)}{\int_{\partial\Sigma} ds_\sigma} = 0, & \text{on } \partial\Sigma, \\ \int_\Sigma e^{-2\psi} dA_\sigma = A. \end{cases}$$

Write this one-to-one correspondence between uniform metrics of types I and II as

$$\begin{aligned} \Psi : \mathbf{M}_{II}(\Sigma, A) &\rightarrow \mathbf{M}_I(\Sigma, A), \quad \Psi(\sigma) = \tau, \\ \Phi : \mathbf{M}_I(\Sigma, A) &\rightarrow \mathbf{M}_{II}(\Sigma, A), \quad \Phi(\tau) = \sigma. \end{aligned}$$

It induces a one-to-one correspondence

$$\tilde{\Psi} : \mathcal{M}_{II}(\Sigma, A) \rightarrow \mathcal{M}_I(\Sigma, A), \quad \tilde{\Phi} : \mathcal{M}_I(\Sigma, A) \rightarrow \mathcal{M}_{II}(\Sigma, A).$$

This correspondence is nothing but a translation of the uniformization theorem of Osgood, Phillips, and Sarnak [O-P-S1] into our context. We have the following theorem whose proof is given in the next section.

Theorem 8.1. *The one-to-one maps Ψ, Φ are continuous and therefore homeomorphisms.*

As a direct corollary, we have:

Theorem 8.2. *The maps $\tilde{\Psi}, \tilde{\Phi}$ are homeomorphisms.*

8.2. Properness of the height on $\mathcal{M}_I(\Sigma, A)$. We can rewrite Corollary 3.4 as the following.

Theorem 8.3. *For each $A > 0$, the height h is a proper function on the moduli space $\mathcal{M}_I(\Sigma, A)$, i.e.*

$$h(M) \rightarrow +\infty$$

as the isometry class $[M]$ approaches $\partial\mathcal{M}_I(\Sigma, A)$.

9. PROOF OF THEOREM 8.1

Throughout this section we extensively use the proof of the uniformization theorem of Osgood, Phillips, and Sarnak (see especially pp.158–163 of [O-P-S1], but note that their Laplacian and our Laplacian have opposite signs). Let $\tau \in \mathbf{M}_I(\Sigma, A)$, $\sigma \in \mathbf{M}_{II}(\Sigma, A)$. Suppose $\Psi(\sigma) = \tau$ (and so $\Phi(\tau) = \sigma$). We assume without loss of generality that the fixed area A is $-2\pi\chi(\Sigma)$ and so the constant curvature K_τ is -1 .

9.1. Continuity of Ψ . First normalize σ by a conformal factor $e^{2\varphi_0}$, where φ_0 is a solution of the following differential equation:

$$\partial_n^\sigma \varphi_0 + k_\sigma = 0 \quad \text{on } \partial\Sigma.$$

The resulting metric $\sigma_0 = e^{2\varphi_0}\sigma$ has vanishing geodesic curvature of the boundary, i.e. $k_{\sigma_0} = 0$. Moreover, it is possible to pick φ_0 so that it (and also σ_0) depends continuously on σ in C^∞ . Construct φ_0 in the following way. Let η be a smooth function compactly supported on the interval $[0, 1]$ and $\eta \equiv 1$ on $[0, 1/2]$. For $s > 0$, define $H_s : \partial\Sigma \rightarrow \Sigma$ by

$$H_s(p) = \exp_p(s\vec{n}), \quad p \in \partial\Sigma,$$

where \exp and \vec{n} are the exponential map and the inward normal vector with respect to the metric σ , respectively. Let

$$r := \sup\{s > 0 \mid H_t \text{ is an embedding for every } t < s\}.$$

On each inward normal geodesic ray emanating from $p \in \partial\Sigma$ of length r , first solve the ODE

$$f_p'(t) + k_\sigma = 0, \quad f_p(0) = 0.$$

Define φ_0 on Σ as

$$\varphi_0(x) = \begin{cases} \eta(s/r)f_p(s), & \text{if } x = H_s(p) \text{ for some } s < r \text{ and } p \in \partial\Sigma; \\ 0, & \text{otherwise,} \end{cases}$$

then it has the desired property.

Without loss of generality, further normalize σ_0 so that it has area A .

Consider the following functional with the reference metric σ_0 :

$$F_1(\varphi) = \frac{1}{2} \int_\Sigma |\nabla_0 \varphi|^2 dA_0 + \int_\Sigma K_0 \varphi dA_0 - \pi\chi(\Sigma) \log \left(\int_\Sigma e^{2\varphi} dA_0 \right).$$

Subject to the constraint

$$(9.1) \quad \int_\Sigma \varphi dA_0 = 0,$$

F_1 is strictly convex and there exists a unique minimizer ψ of F_1 such that after a proper rescaling, the metric $\tau = e^{2\psi}\sigma_0 = \Psi(\sigma)$ is in $\mathcal{M}_I(\Sigma, A)$ (see [O-P-S1]). By the first variation of F_1 , we see that ψ satisfies

$$(9.2) \quad \Delta_0 \psi = -K_0 + \frac{2\pi\chi(M)}{\int_M e^{2\psi} dA_0} e^{2\psi} \quad \text{in } \Sigma,$$

$$(9.3) \quad \partial_n \psi = 0 \quad \text{on } \partial\Sigma$$

(see [O-P-S1] pp161). For the continuity of Ψ , we show next that ψ depends continuously on σ in C^∞ .

Suppose there is a sequence of metrics $\{\sigma^i\}$ such that $\sigma^i \rightarrow \sigma$ in C^∞ . Consider the corresponding normalized metrics σ_0^i constructed as above, and see that $\sigma_0^i \rightarrow \sigma_0$ in C^∞ . Let M_i (resp. M) denote the Riemannian manifold (Σ, σ_0^i) (resp. (Σ, σ_0)), and let F_1^i be the corresponding functional as above, and ψ^i be the corresponding unique minimum, subject to the constraint (9.1) with respect to σ_0^i .

We now show that $\psi^i \rightarrow \psi$ in C^∞ . Several steps are similar (sometimes verbatim) to the arguments given in [O-P-S1]. In the following steps, if not specified, the metric related quantities such as measure or gradient inside the integral \int_{M_i} (resp. \int_M) will be those of the corresponding metric σ_0^i (resp. σ_0); it is also important to keep in mind that all these metric related quantities are comparable with each other, respectively, because $\sigma_0^i \rightarrow \sigma_0$ in C^∞ .

Step 1. Since $\sigma_0^i \rightarrow \sigma_0$ in C^∞ , it is clear that

$$(9.4) \quad F_1^i(\psi^i) \leq F_1^i(0) \leq \text{const..}$$

Step 2: Obtain a priori bounds of ψ^i . By Poincaré inequality (with the constraint (9.1)),

$$\left| \int_{M_i} K_0^i \psi^i \right| \lesssim \left(\int_{M_i} |\psi^i|^2 \right)^{1/2} \lesssim \left(\int_{M_i} |\nabla \psi^i|^2 \right)^{1/2}.$$

By Jensen's inequality and noting $\int_{M_i} \psi^i = 0$,

$$\log \left(\int_{M_i} \exp(2\psi^i) \right) \geq \log A.$$

Combining these inequalities with (9.4), we get

$$\int_{M_i} |\nabla \psi^i|^2 \leq \text{const..}$$

Since $\sigma_0^i \rightarrow \sigma_0$ in C^∞ , we also have

$$(9.5) \quad \int_M |\nabla \psi^i|^2 \leq \text{const..}$$

Step 3: $\psi^i \rightarrow \psi$ in the space $L^2(M)$. (9.5) implies

$$\int_M |\psi^i|^2 \lesssim \int_M |\nabla \psi^i|^2 \leq \text{const..}$$

By Rellich's compactness, there exists ψ^∞ in the Sobolev space $W^1(M)$ such that a subsequence $\psi^{i_k} \rightarrow \psi^\infty$ weakly in $W^1(M)$, strongly in $L^2(M)$, and pointwise a.e.

Therefore

$$\begin{aligned} \int_M |\nabla \psi^\infty|^2 &\leq \liminf \int_{M_{i_k}} |\nabla \psi^{i_k}|^2, \\ \int_M K_0 \psi^\infty &= \lim \int_{M_{i_k}} K_0^{i_k} \psi^{i_k}, \\ \int_M \exp(2\psi^\infty) &\leq \liminf \int_{M_{i_k}} \exp(2\psi^{i_k}), \end{aligned}$$

hence

$$(9.6) \quad F_1(\psi^\infty) \leq \liminf F_1^{i_k}(\psi^{i_k}).$$

Let $\epsilon_k = \int_{M_{i_k}} \psi$, $\psi_{\epsilon_k} = \psi - \frac{\epsilon_k}{A}$, therefore $\int_{M_{i_k}} \psi_{\epsilon_k} = 0$. Obviously $F_1^{i_k}(\psi^{i_k}) \leq F_1^{i_k}(\psi_{\epsilon_k})$, and $F_1^{i_k}(\psi_{\epsilon_k}) \rightarrow F_1(\psi)$. Then by (9.6) we have

$$F_1(\psi^\infty) \leq F_1(\psi).$$

This implies $\psi^\infty = \psi$ by the uniqueness of the F_1 minimum. Since this convergence holds for every convergent subsequence, we conclude that

$$(9.7) \quad \psi^i \rightarrow \psi \quad \text{in } L^2(M).$$

Step 4. For convergence in higher Sobolev spaces, the elliptic regularity of (9.2)(9.3) is used. First, there is an *a priori* bound

$$(9.8) \quad \|\psi^i\|_t \leq \text{const.}_t$$

for each ($t > 0$) Sobolev t -norm $\|\cdot\|_t$ on M . (Trudinger's inequality is used when we bound the righthand-side of (9.2) for ψ^i in $\|\cdot\|_0$. See [O-P-S1]: equations (3.19) and (3.24), and the remark below (3.29) in it.) By Rellich's compactness and a diagonal argument, we see from (9.7), (9.8) that

$$\|\psi^i - \psi\|_t \rightarrow 0 \quad \text{for each } t > 0.$$

This completes the proof of the C^∞ convergence of ψ^i to ψ , and so the continuity of Ψ .

9.2. Continuity of Φ . A similar (sometimes verbatim) argument as in the case of Ψ is used.

First find a solution for

$$\Delta_\tau \varphi_0 = 1 \text{ in } \Sigma,$$

so the resulting metric $\tau_0 = e^{\varphi_0} \tau$ is flat, i.e. $K_{\tau_0} = 0$, and depends continuously on τ in C^∞ . φ_0 can be constructed in the following way. Double Σ with metric τ to get a closed hyperbolic surface $\tilde{\Sigma}$. Σ is one half of $\tilde{\Sigma}$. Let η be a smooth function compactly supported on the interval $[0, 1)$ and $\eta \equiv 1$ on $[0, 1/2]$. For $s > 0$, define $H_s : \partial\Sigma \rightarrow \tilde{\Sigma}$ by

$$H_s(p) = \exp_p(s\vec{n}), \quad p \in \partial\Sigma,$$

where \exp , \vec{n} are the exponential map and the outward normal vector (for $\partial\Sigma$) with respect to τ , respectively. Let

$$r := \sup\{s > 0 \mid H_t \text{ is an embedding for every } t < s\}.$$

Define f_0 on $\tilde{\Sigma}$ as

$$f_0(x) = \begin{cases} 1, & \text{if } x \in \Sigma; \\ \eta(s/r), & \text{if } x = H_s(p) \text{ for some } s < r \text{ and } p \in \partial\Sigma; \\ 0, & \text{otherwise.} \end{cases}$$

Now fix $\delta > 0$ and a point p outside the $r/2$ -neighborhood U of Σ so that the δ -geodesic ball of p does not intersect U . Pick a radially symmetric C^∞ bump function, say f_1 , supported on this δ -geodesic ball so that the function $f = f_0 - f_1$ satisfies $\int_\Sigma f dA_\tau = 0$. Then find the solution for $\tilde{\varphi}_0$ (uniquely up to constant) where

$$\Delta_\tau \tilde{\varphi}_0 = f \text{ on } \tilde{\Sigma}.$$

By restricting $\tilde{\varphi}_0$ to Σ we find the desired function φ_0 .

From this point on, without loss of generality, further rescale the flat metric τ_0 so that it has area A .

Consider the following functional with the reference metric τ_0 .

$$F_2(\varphi) = \frac{1}{2} \int_\Sigma |\nabla_0 \varphi|^2 dA_0 + \int_{\partial\Sigma} k_0 \varphi ds_0 - 2\pi\chi(\Sigma) \log \left(\int_{\partial\Sigma} e^\varphi ds_0 \right),$$

Subject to the constraint

$$(9.9) \quad \int_{\partial\Sigma} \varphi ds_0 = 0,$$

F_2 is strictly convex and there exists a unique minimizer ϕ of F_2 so that the metric $e^{2\phi}\tau_0$ is a uniform metric of type II [O-P-S1]. In particular, ϕ is a harmonic function with respect to the flat metric τ_0 . We obtain $\sigma = \Phi(\tau)$ by rescaling this metric (i.e. adding a constant to ϕ) so that it has the area A . Let T be the Dirichlet-Neumann operator of the metric τ_0 , i.e. T is the linear operator on functions on $\partial\Sigma$ given by

$$T\varphi = \partial_n \tilde{\varphi}$$

where $\tilde{\varphi}$ is the harmonic extension of φ into Σ . As in [O-P-S1], T is a positive self-adjoint pseudodifferential operator of order 1 on the space of functions φ with 9.9, and we have

$$(9.10) \quad |T^{1/2}\varphi|_0 \sim |\varphi|_{1/2} \quad \text{and} \quad |T\varphi|_0 \sim |\varphi|_1,$$

where $|\cdot|_t$ is the Sobolev t -norm on $\partial\Sigma$ with respect to the metric τ_0 . Now as in [O-P-S1] the minimizer ϕ of F_2 satisfies

$$(9.11) \quad T\phi = -k_0 + \frac{2\pi\chi(\Sigma)e^\phi}{\int_{\partial\Sigma} e^\phi ds_0},$$

and from harmonicity of ϕ ,

$$(9.12) \quad F_2(\phi) = \frac{1}{2} \int_{\partial\Sigma} \phi T \phi ds_0 + \int_{\partial\Sigma} k_0 \phi ds_0 - 2\pi\chi(\Sigma) \log \left(\int_{\partial\Sigma} e^\phi ds_0 \right),$$

$$(9.13) \quad = \frac{1}{2} |T^{1/2} \phi|_0^2 + \int_{\partial\Sigma} k_0 \phi ds_0 - 2\pi\chi(\Sigma) \log \left(\int_{\partial\Sigma} e^\phi ds_0 \right).$$

By the expression of (9.13), the functional F_2 induces a functional on the space of functions on $\partial\Sigma$. $\phi|_{\partial\Sigma}$ is the unique minimizer of this induced functional subject to the constraint (9.9).

For the continuity of Φ , we show that ϕ depends continuously on τ in C^∞ . Our plan is first to derive this continuous dependence for the restriction $\phi|_{\partial\Sigma}$ to the boundary and then use harmonicity of ϕ to obtain the continuous dependence for the whole function ϕ .

Suppose there is a sequence of metrics $\{\tau^i\}$ such that $\tau^i \rightarrow \tau$ in C^∞ . Consider the normalized flat metrics τ_0^i , constructed as above, and see that $\tau_0^i \rightarrow \tau_0$. Let M_i (resp. M) be the Riemannian manifold (Σ, τ_0^i) (resp. (Σ, τ_0)). Let F_2^i be the corresponding functional as above; and let ϕ^i be the corresponding unique minimum subject to the constraint (9.9) with respect to τ_0^i . To show that $\phi^i \rightarrow \phi$ in C^∞ , several steps which are similar (sometimes verbatim) to the arguments given in [O-P-S1] are taken. In the following, if not specified, the metric related quantities such as measure, gradient, or T inside the integrals $\int_{M_i}, \int_{\partial M_i}$ (resp. $\int_M, \int_{\partial M}$) will be those of the corresponding metric τ_0^i (resp. τ_0); it is also important to keep in mind that all these metric related quantities are all comparable with each other, respectively, because $\tau_0^i \rightarrow \tau_0$ in C^∞ .

Step 1. Since $\tau_0^i \rightarrow \tau_0$ in C^∞ , it is clear that

$$(9.14) \quad F_2^i(\phi^i) \leq F_2^i(0) \leq \text{const..}$$

Step 2. Use Step 1, to obtain some *a priori* bounds of ϕ^i . By (9.10),

$$\left| \int_{\partial M_i} k_0^i \phi^i \right| \lesssim \left(\int_{\partial M_i} |\phi^i|^2 \right)^{1/2} \lesssim \left(\int_{\partial M_i} |T^{1/2} \phi^i|^2 \right)^{1/2}.$$

By Jensen's inequality and noting $\int_{\partial M_i} \phi^i = 0$,

$$\log \left(\int_{\partial M_i} \exp(\phi^i) \right) \geq \log \int_{\partial M_i} 1 \geq \text{const..}$$

Combine these inequalities with (9.13) and (9.14), we get

$$\int_{\partial M_i} |T^{1/2} \phi^i|^2 \leq \text{const..}$$

Since $\tau_0^i \rightarrow \tau_0$ in C^∞ , we also have

$$(9.15) \quad \int_{\partial M} |T^{1/2} \phi^i|^2 \leq \text{const..}$$

Step 3. In this step, we show $\phi^i \rightarrow \phi$ in the space $L^2(\partial M)$. (9.15) combined with (9.10) implies

$$\int_{\partial M} |\phi^i|^2 \lesssim \int_{\partial M} |T^{1/2}\phi^i|^2 \leq \text{const.}$$

By Rellich's compactness there is ϕ^∞ in the Sobolev space $W^{1/2}(\partial M)$ such that a subsequence $\phi^{i_k} \rightarrow \phi^\infty$ weakly in $W^{1/2}(\partial M)$ but strongly in $L^2(\partial M)$, and pointwise a.e. This results

$$\begin{aligned} \int_{\partial M} |T^{1/2}\phi^\infty|^2 &\leq \liminf \int_{\partial M_{i_k}} |T^{1/2}\phi^{i_k}|^2, \\ \int_{\partial M} k_0\phi^\infty &= \lim \int_{\partial M_{i_k}} k_0^{i_k}\phi^{i_k}, \\ \int_{\partial M} \exp(\phi^\infty) &\leq \liminf \int_{\partial M_{i_k}} \exp(\phi^{i_k}), \end{aligned}$$

therefore

$$(9.16) \quad F_2(\phi^\infty) \leq \liminf F_2^{i_k}(\phi^{i_k}).$$

Denote $\epsilon_k = \int_{\partial M_{i_k}} \phi$ and let $\phi_{\epsilon_k} = \phi - \epsilon_k / (\int_{\partial M_{i_k}} 1)$ so that $\int_{\partial M_{i_k}} \phi_{\epsilon_k} = 0$. Then obviously $F_2^{i_k}(\phi^{i_k}) \leq F_2^{i_k}(\phi_{\epsilon_k})$ and $F_2^{i_k}(\phi_{\epsilon_k}) \rightarrow F_2(\phi)$. From (9.16),

$$F_2(\phi^\infty) \leq F_2(\phi).$$

This implies $\phi^\infty = \phi$ by the uniqueness of the minimum of F_2 . Since this convergence is for every convergent subsequence, we conclude that

$$(9.17) \quad \phi^i \rightarrow \phi \quad \text{in } L^2(\partial M).$$

Step 4. For convergence in higher Sobolev spaces $W^t(\partial M)$, $t > 0$, we use the elliptic regularity of (9.11). We need the following lemma.

Lemma 9.1. (See [O-P-S3]: eqn. (2.10) and Lemma 2.5. See also [O-P-S1]: Lemma 3.5 and Lemma 3.9.) *For a fixed flat metric on Σ , there exist positive constants such that for any smooth function f on Σ ,*

$$(9.18) \quad |f|_{1/2}^2 \leq \text{const.} \left\{ \int_{\Sigma} |\nabla f|^2 dA + \left(\int_{\partial\Sigma} f ds \right)^2 \right\},$$

$$(9.19) \quad \int_{\partial\Sigma} e^f \frac{ds}{\int_{\partial\Sigma} ds} \leq \text{const.} \exp \left(\text{const.} \int_{\Sigma} |\nabla f|^2 dA + \frac{\int_{\partial\Sigma} f ds}{\int_{\partial\Sigma} ds} \right).$$

In particular,

$$(9.20) \quad \int_{\partial\Sigma} e^f \frac{ds}{\int_{\partial\Sigma} ds} \leq \text{const.} \exp \left(\text{const.} |T^{1/2}f|_0^2 + \frac{\int_{\partial\Sigma} f ds}{\int_{\partial\Sigma} ds} \right).$$

First, it follows that

$$(9.21) \quad |\phi^i|_t \leq \text{const.}_t \quad \text{for each } t > 0.$$

We have used (9.15) and (9.20) when we bound the righthand-side of (9.11) for ϕ^i in $|\cdot|_0$.

As in the Step 4 for the case of Ψ , apply Rellich's compactness and a diagonal argument to (9.17) and (9.21) to get

$$(9.22) \quad |\phi^i - \phi|_t \rightarrow 0 \text{ for each } t > 0.$$

Step 5. By elliptic regularity theory, observe the following standard fact.

Lemma 9.2. *If*

$$\Delta f = 0 \text{ in } \Sigma \quad \text{and} \quad |f|_t < \infty \text{ on } \partial\Sigma,$$

then

$$\|f\|_t \leq \text{const.}_t |f|_{t-1/2} \leq \text{const.}_t |f|_t.$$

Apply this lemma to ϕ^i to get $\|\phi^i\|_t \leq \text{const.}_t$. By Rellich's compactness and a diagonal argument, there is a subsequence ϕ^{i_k} converging to some function ϕ^∞ on Σ in C^∞ . This ϕ^∞ is harmonic for the metric τ_0 ; and by (9.22), $\phi^\infty|_{\partial\Sigma} = \phi|_{\partial\Sigma}$. Therefore by uniqueness of Dirichlet problem we see $\phi^\infty = \phi$ in Σ . This is for every convergent subsequence and so we conclude that $\phi^i \rightarrow \phi$ on Σ in C^∞ . The proof of the continuity of Φ is thus complete.

10. COMPACTNESS OF THE SET OF ISOSPECTRAL FLAT SURFACES WITH BOUNDARY

In this section we prove the following theorem.

Theorem 10.1. *Let Σ be a compact orientable surface with boundary. Assume $\chi(\Sigma) < 0$. Then each Dirichlet isospectral set of isometry classes of flat metrics on Σ is compact in the C^∞ -topology.*

Let $\{\rho_i\}_{i=1}^\infty$ be a sequence of flat (i.e. sectional curvature is zero) metrics on Σ having the same Dirichlet spectrum. By heat kernel asymptotic expansions (see [M-S]), all such metrics ρ_i 's have the same area, say A , as well as the same boundary length, say L . Write $\rho_i = e^{2\phi_i}\sigma_i$ where σ_i is a uniform metric of type II of area A . Each σ_i induces an element $[\sigma_i] \in \mathcal{M}_{II}(\Sigma, A)$.

Proposition 10.2. *The above sequence $\{[\sigma_i]\}$ is compact in $\mathcal{M}_{II}(\Sigma, A)$.*

Proof. We drop the index i for a moment. Consider $\tau = \Psi(\sigma)$, i.e. τ is a uniform metric of type I of area A , and write $\rho = e^{2\psi}\tau$. Without loss of generality we may assume $A = -2\pi\chi(\Sigma)$ and so the curvature K_τ of τ is -1 . Polyakov-Alvarez formula reads

$$(10.1) \quad h(e^{2\psi}\tau) = \frac{1}{6\pi} \left\{ \frac{1}{2} \int_\Sigma |\nabla_\tau \psi|^2 dA_\tau - \int_\Sigma \psi dA_\tau \right\} + \frac{1}{4\pi} \int_{\partial\Sigma} \partial_n^\tau \psi ds_\tau + h(\tau)$$

Since by (7.1) $\Delta_\tau \psi = 1$, we see

$$\int_{\partial\Sigma} \partial_n^\tau \psi ds_\tau = - \int_\Sigma \Delta_\tau \psi dA_\tau = 2\pi\chi(\Sigma)$$

and by Jensen's inequality

$$\int_{\Sigma} 2\psi dA_{\tau}/A \leq \log \left(\int_{\Sigma} e^{2\psi} dA_{\tau}/A \right) = \log 1 = 0.$$

So back to (10.1) we get

$$(10.2) \quad h(e^{2\psi}\tau) \geq \frac{\chi(\Sigma)}{2} + h(\tau).$$

For each ρ_i , consider the corresponding uniform metric τ_i of type I, i.e. $\tau_i = \Psi(\sigma_i)$. By isospectrality, $h(\rho_i) = \text{const.}$ for all i . Thus by (10.2), $h(\tau_i) \leq \text{const.}$ From Theorem 8.3, h is proper and bounded below on $\mathcal{M}_I(\Sigma, A)$, and so the sequence $\{\tau_i\}$ induced in $\mathcal{M}_I(\Sigma, A)$ is compact. Apply the map $\tilde{\Phi}$ (see Theorem 8.2), to see $\{\sigma_i\}$ is compact. \square

Remark 10.1. It seems hard to get directly the compactness of the type II uniformization of isospectral flat metrics, without working on the type I uniformization first then using Theorem 8.1 as we did in the above proof. The technical difficulty exists because the corresponding Polyakov-Alvarez formula has the term involving boundary geodesic curvature of the type II uniform metrics, which we do not have good control of when the area is fixed. This is true especially if the metrics are near the boundary of the corresponding moduli space, a case *a priori* possible though excluded by Proposition 10.2.

From Proposition 10.2, we can find isometric representatives of σ_i 's which consist a compact set in C^∞ , and we denote these representatives by the same σ_i 's. The remaining part is in principle the same as in the last section of [O-P-S3]. It is included for reader's convenience. Note that by compactness of $\{\sigma_i\}$, all the Sobolev t -norms $\|\cdot\|_t$ of functions on Σ , resp. $|\cdot|_t$ of functions on $\partial\Sigma$, resp. the C^j norms, induced by σ_i are equivalent by uniformly bounded constant multiples; therefore, in the consideration below, we may without loss of generality deal with only such norms for one fixed metric. Note also that all the quantities induced by the metric σ_i are uniformly bounded in the C^j norm for each j . Now, for compactness of the (isometric representative) sequence $\{\rho_i\}$, it is enough to show that $\{\phi_i\}$ has a convergent subsequence in C^∞ . By a diagonal argument using Rellich's compactness it is equivalent to show that the sequence $\{\phi^i\}$ is uniformly bounded in $\|\cdot\|_t$ for each integer t .

From this point on, we will drop the index i . Using (7.1) and (7.2), we see

$$(10.3) \quad \Delta_{\sigma}\phi = 0,$$

$$(10.4) \quad k = e^{-\phi}(k_{\sigma} + \partial_n^{\sigma}\phi)$$

where $k_{\sigma} = \frac{2\pi\chi(\Sigma)}{\int_{\partial\Sigma} ds_{\sigma}}$ and ∂_n^{σ} is now the Dirichlet-Neumann operator. By Lemma 9.2, it is enough to show that the ϕ 's are uniformly bounded for $|\cdot|_t$ for each integer t . We will do an induction on t .

Step 1[O-P-S3] First let's deal with the Sobolev 1-norm of ϕ . The Polyakov-Alvarez formula reads

$$(10.5) \quad h(e^{2\phi}\sigma) = \frac{1}{6\pi} \left\{ \frac{1}{2} \int_{\Sigma} |\nabla_{\sigma}\phi|^2 dA_{\sigma} + k_{\sigma} \int_{\partial\Sigma} \phi ds_{\sigma} \right\} + h(\sigma).$$

By isospectrality,

$$(10.6) \quad h(e^{2\phi}\sigma) = \text{const.},$$

and $h(\sigma)$ is bounded above and below by the compactness of the set $\{\sigma_i\}$. Since $\int_{\partial\Sigma} e^{\phi} ds_{\sigma} = L$, by Jensen's inequality

$$\int_{\partial\Sigma} \phi ds_{\sigma} \leq \log\left(\frac{L}{\int_{\partial\Sigma} ds_{\sigma}}\right) \int_{\partial\Sigma} ds_{\sigma} \leq \text{const..}$$

Combining this with (10.5), (10.6), and the fact that $k_{\sigma} < 0$ and $|k_{\sigma}|$ is bounded, we see

$$(10.7) \quad |k_{\sigma} \int_{\partial\Sigma} \phi ds_{\sigma}| \leq \text{const.}$$

and

$$(10.8) \quad \left| \int_{\partial\Sigma} \phi ds_{\sigma} \right| \leq \text{const..}$$

By (10.5), (10.6), and (10.7),

$$(10.9) \quad \int_{\Sigma} |\nabla_{\sigma}\phi|^2 dA_{\sigma} \leq \text{const..}$$

By (10.8), (10.9), (9.18), and (9.19),

$$(10.10) \quad |\phi|_{1/2} \leq \text{const.},$$

$$(10.11) \quad \int_{\partial\Sigma} e^{\phi} ds_{\sigma}, \quad \int_{\partial\Sigma} e^{2\phi} ds_{\sigma} \leq \text{const..}$$

By Melrose's result [Me] for an isospectral set of flat metrics, the geodesic curvature k as a function on $\partial\Sigma$ is uniformly bounded in the C^j norm for each j . (His original result was in Euclidean context but it can be easily carried over to a flat surface case.) Since k and k_{σ} are uniformly bounded, we see by (10.4),

$$|\partial_n^{\sigma}\phi|^2 \leq \text{const.}(e^{2\phi} + e^{\phi} + 1),$$

and (10.11) gives

$$(10.12) \quad \int_{\partial\Sigma} |\partial_n^{\sigma}\phi|^2 ds_{\sigma} \leq \text{const.} \left(\int_{\partial\Sigma} e^{2\phi} ds_{\sigma} + \int_{\partial\Sigma} e^{\phi} ds_{\sigma} + \int_{\partial\Sigma} ds_{\sigma} \right) \leq \text{const..}$$

Lemma 10.3. (See [O-P-S3]: eqn. (5.3)) *For a smooth function f on $\partial\Sigma$,*

$$|\partial_n f|_0 + |f|_0 = |f|_1,$$

where ∂_n is the Dirichlet-Neumann operator.

So by (10.10) and (10.12) we get the Sobolev 1-norm $|\cdot|_t$ bound for ϕ . In particular, the ϕ 's are uniformly bounded on $\partial\Sigma$.

Step 2[O-P-S3] From (10.4), for the derivative ∂_s along $\partial\Sigma$

$$|\partial_s^t \partial_n \phi|_0 = |\partial_s^t (-k_\sigma + e^\phi k)|_0 \leq \text{const.}_t |\phi|_t.$$

The Dirichlet-Neumann operator ∂_n is a pseudodifferential operator of order 1 and

$$|[\partial_s^t, \partial_n] \phi|_0 \leq \text{const.}_t |\phi|_t,$$

where $[\cdot, \cdot]$ is the commutator operation. Now by Lemma 10.3,

$$|\phi|_{t+1} \leq |\partial_n \partial_s^t \phi|_0 + |\phi|_t \leq \text{const.}_t |\phi|_t.$$

This completes the induction and finishes the proof of the C^∞ -compactness of the set $\{\phi_i\}$ and so of $\{\rho_i\}$, and so finalizes the proof of Theorem 10.1.

11. PROPERNESS OF THE HEIGHT ON $\mathcal{M}_{II}(\Sigma, A)$

Let $\sigma \in \mathbf{M}_{II}(\Sigma, A)$ and $\tau \in \mathbf{M}_I(\Sigma, A)$ such that $\sigma = e^{2\psi} \tau$. Proceed exactly as for (10.2) to get

$$(11.1) \quad h(e^{2\psi} \tau) \geq \frac{1}{2} \chi(\Sigma) + h(\tau).$$

This allows us to show the following theorem.

Theorem 11.1. *For each $A > 0$, the height h is a proper function on the moduli space $\mathcal{M}_{II}(\Sigma, A)$, i.e.*

$$h(M) \rightarrow +\infty$$

as the isometry class $[M]$ approaches $\partial\mathcal{M}_{II}(\Sigma, A)$.

Proof. The theorem follows from (11.1), Theorem 8.2, and Theorem 8.3. \square

Remark 11.1. It is easy to see that for two conformally equivalent uniform metrics τ and σ , respectively of type I and II, of the same boundary length,

$$h(\tau) \geq \frac{1}{2} \chi(\Sigma) + h(\sigma).$$

12. FURTHER REMARKS

It is desirable to find a connection between our results and the results of Khuri [Kh], Osgood, Phillips, and Sarnak [O-P-S3]: they showed the properness (resp. non-properness) of height function [O-P-S3] (resp. [Kh]) for $(0, n)$ -type surfaces, $n > 0$ (resp. for (g, n) -type surfaces, $g > 0, n > 0$), when the boundary length is fixed instead of the area. Note that the two conditions, one fixes the area and the other fixes the boundary length, are equivalent by scaling, and there is a nice formula (7.4) for the change of height under scaling.

It seems interesting to get a better understanding of the maps $\tilde{\Psi}$ and $\tilde{\Phi}$ between the two spaces $\mathcal{M}_I(\Sigma, A)$ and $\mathcal{M}_{II}(\Sigma, A)$. For example, it would be nice to investigate the extension of these maps to the compactifications of $\mathcal{M}_I(\Sigma, A)$ and $\mathcal{M}_{II}(\Sigma, A)$, and it would also be interesting to see the geometric properties of these maps with respect to certain natural metrics given on those moduli spaces.

Finally, we wonder whether the set of isospectral Riemannian metrics (without any further restriction) on a compact surface with boundary (or a higher dimensional manifold) is compact in the C^∞ -topology. One may try to modify the method of Osgood, Phillips, and Sarnak in [O-P-S2].

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