DETERMINANTS OF LAPLACIANS, QUASIFUCHSIAN SPACES, AND HOLOMORPHIC EXTENSIONS OF LAPLACIANS

YOUNG-HEON KIM

Abstract. The Teichmüller space $Teich(S)$ of a surface S in genus $g > 1$ is reallized as a totally real submanifold of the quasifuchsian space $QF(S)$. We show that the determinant of the Laplacian $\det'(\Delta)$ on $Teich(S)$ has a unique holomorphic extension to $QF(S)$. To realize this holomorphic extension as the determinant of differential operators on S, we introduce a holomorphic family $\{\Delta_{\mu,\nu}\}$ of elliptic second order differential operators on S whose parameter space is the space of pairs of Beltrami differentials on S and which naturally extends the Laplace operators of hyperbolic metrics on S. We study the determinant of this family $\{\Delta_{\mu,\nu}\}$ and show how this family realizes the holomorphic extension of $\det'(\Delta)$ as its determinant.

In this note, we present the results from [Ki] to which we refer for details.

Let X be a compact Riemann surface of genus $g > 1$, and let Δ be the Laplacian on scalar functions on X, which extends to $L^2(X)$ with respect to the hyperbolic metric, i.e. on the universal cover $\mathbb{H}$ of X, the pull-back of Δ by the covering map is the hyperbolic Laplacian

$$\Delta_{\mathbb{H}} = (z - \overline{z})^2 \frac{\partial^2}{\partial z \partial \overline{z}}.$$

The Laplacian Δ has eigenvalues $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty$, and the determinant of the operator Δ may be defined formally as the product of the nonzero eigenvalues of Δ. A regularization $\det'(\Delta)$ of this product was defined by Ray and Singer [RS1] [RS2], using the zeta function of Δ:

$$\zeta_\Delta(s) = \sum_{\lambda \in \mathrm{Spec}(\Delta)\setminus\{0\}} \lambda^{-s}.$$

This infinite sum is absolutely convergent for $\mathrm{Re}\, s > 1$, and has a meromorphic extension to the whole complex plane which is regular at $s = 0$, and the determinant $\det'(\Delta)$ is defined (see [RS1]) as

$$-\log \det'(\Delta) = \frac{d\zeta_\Delta(0)}{ds}.$$

This determinant $\det'(\Delta)$ has appeared to be very important in mathematics. For example, in [OPS1], (see also [Sa2]), Osgood, Phillips and Sarnak studied $-\log\det'(\Delta)$ as a "height" function on the space of metrics on a compact orientable smooth surface S of genus g. For $g > 1$, they showed that when restricted to a given conformal class of metrics on S, it attains its minimum at the unique hyperbolic metric in this conformal class, and has no other critical points. Thus, to find Riemannian metrics on S which are extremal, in the

Date: February 27, 2006.

2000 *Mathematics Subject Classification.* 32G15; 58J52.

This research was partially supported by the NSF under grant DMS-0072508.

sense that they minimize $-\log\det'(\Delta)$, it suffices to consider its restriction to the moduli space $\mathcal{M}_g$ of hyperbolic metrics on a Riemann surface S of genus g. It was shown by Wolpert that this restriction is a proper function (see [W2]), which was used also by Osgood, Phillips and Sarnak to show that the isospectral sets (with respect to the Laplacian) of isometry classes of metrics on S are all compact in the C^∞ topology (see [OPS2]). The determinant of the Laplacian also appears in string theory, especially when we renormalize Feynmann integrals over spaces of surfaces (see, for example, [Po] [BK]).

The universal cover of the orbifold $\mathcal{M}_g$, with covering group the mapping class group Γ_g, is the Teichmüller space *Teich*(S) which is biholomorphic to a bounded open domain in the complex space $\mathbb{C}^{3g-3}$. The function $-\log\det'(\Delta)$ lifts to a function on the Teichmüller space *Teich*(S) invariant under Γ_g. Our first result concerns the function theoretic properties of $\log\det'(\Delta)$ on *Teich*(S).

0.1. **Holomorphic extensions of determinants of Laplacians.** Let's consider the special case of genus 1.

Example ([RS2] or [Sa1], p. 33, (A.1.7)). For $z \in \mathbb{H}$, let T_z be the flat torus obtained by the lattice of $\mathbb{C}$ generated by 1 and z. Then the determinant of Laplacian of this flat torus is

$$\log\det'(\Delta)(z) = \log(2\pi(\mathrm{Im}\ z)^{1/2}|\eta(z)|^2)$$

where $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ for $q = e^{2\pi i z}$ is the Dedekind eta function.

The manifold $\mathbb{H}$ has a complexification $\mathbb{H}\times\overline{\mathbb{H}}$, and the function $\log\det'(\Delta)(z)$ on the diagonal $\{w = \overline{z}\}$ has a unique holomorphic extension to $\mathbb{H}\times\overline{\mathbb{H}}$, namely,

$$\log\bigl(2\pi(\frac{z-w}{2i})^{1/2}\,\eta(z)\,\overline{\eta(\overline{w})}\bigr).$$

We show that even in higher genus $g > 1$, the function $\log\det'(\Delta)$ has a unique holomorphic extension. In higher genus, the objects corresponding to $\mathbb{H}$ and $\mathbb{H}\times\overline{\mathbb{H}}$ are the Teichmüller space *Teich*(S) and the quasifuchsian space

$$QF(S) = \mathit{Teich}(S) \times \mathit{Teich}(\overline{S}) \cong \mathit{Teich}(S) \times \overline{\mathit{Teich}(S)},$$

respectively where the real analytic manifold *Teich*(S) imbeds as the diagonal in $QF(S)$. McMullen recently used the quasifuchsian space to study the geometry of the Teichmüller space via the above complexification [Mc].

Theorem 0.1. *The function* $\log\det'(\Delta)$ *on Teich*(S) *has a unique holomorphic extension to the quasifuchsian space* $QF(S)$.

Remark. Historically, the first result in the spirit of Theorem 0.1 is due to Fay [Fa] who obtained a holomorphic extension of the analytic torsion from the Picard variety of a compact Riemann surface to the space of $\mathbb{C}^*$-representations of its fundamental group.

Remark. We note that the holomorphic extension of $\log\det'(\Delta_n)$ of the Laplacian acting on the $(n,0)$-forms for $n \geq 2$ is given by McIntyre and Teo [TM] using the holomorphic extension of Selberg's zeta function. Their method does not work in our case of $\log\det'(\Delta)$ $=\log\det'(\Delta_0)= \log\det'(\Delta_1)$.

Proof of Theorem 0.1: a sketch We use the Belavin-Knizhnik formula (see, for example, [BK] [W1] [ZT]). We only need the following special case of this theorem ([ZT], Theorem 2) that on $Teich(S)$

$$\partial\overline{\partial}\,\log\left(\frac{\det'(\Delta)}{\det(\operatorname{Im}\tau)}\right) = -\frac{i}{6\pi}\,\omega_{WP},$$

where $\operatorname{Im}\tau$ is the imaginary part of the period matrix τ. The differential operator $\partial\overline{\partial}$ comes from the complex structure on $Teich(S)$.

By a result of Platis ([Pl], Theorems 6 and 8), the differential form $i\omega_{WP}$ on the Teichmüller space $Teich(S)$ has an extension to a holomorphic non-degenerate closed $(2,0)$-form Ω on the quasifuchsian space $QF(S)$.

We will construct a holomorphic function q on $QF(S) \cong Teich(S) \times \overline{Teich(S)}$ solving the differential equation $\partial_z\partial_{\overline{w}}q = \Omega$ which extends

$$\partial\overline{\partial}q = i\omega_{WP}.$$

Choose a smooth polar coordinate system on $Teich(S)$ and denote the center of this coordinate system by z_0. Denote the radial line in polar coordinates from z_0 to the point $z \in Teich(S)$ by $\mathsf{v}(z)$. Define $q(z,\overline{w})$ by the formula

$$q(z,w) = \int_{\mathsf{v}(z)\times\overline{\mathsf{v}(w)}} \Omega.$$

The holomorphic extension has the form

$$\log\det'(\Delta)(z,\overline{w}) = -\frac{1}{6\pi}q(z,\overline{w}) + \log\det((\tau(z)-\overline{\tau}(w))/2i) + f(z) + \overline{f}(w),$$

for some holomorphic function f on $Teich(S)$. $\square$

A complex projective ($\mathbb{CP}^1$-)structure on X is a subatlas of charts whose transition functions are in $\mathrm{PSL}(2;\mathbb{C})$. The space $\mathrm{Proj}(S)$ of projective structures on a surface S is naturally a $3g-3$ dimensional complex affine fiber bundle over Teichmüller space $Teich(S)$, which contains the quasifuchsian space $QF(S)$ as an open subset. Each fiber $\mathrm{Proj}_X(S)$ of $\mathrm{Proj}(S)$ over $X \in Teich(S)$ is realized as the banach space of holomorphic quadratic differentials on X with bounded L^∞-norm with respect to the hyperbolic metric on X. The portion of $QF(S)$ in $\mathrm{Proj}_X(S)$ is the image of $Teich(S)$ under the Bers embedding, which is a bounded open domain in $\mathrm{Proj}_X(S)$. See [Mc].

Further along Theorem 0.1, the following natural question seems interesting, which we would like to address in the future.

Question. Does the determinant $\det'(\Delta)$ of the Laplacian holomorphically extend to $\mathrm{Proj}(S)$?

0.2. **Holomorphic extensions of Laplacians.** Another natural question following Theorem 0.1 is whether there is an actual family of elliptic differential operators on S whose determinant realizes the holomorphic extension of $\det'(\Delta)$. To address this question we introduce a family $\{\Delta_{\mu,\nu}\}$ of elliptic second order differential operators on S which is holomorphic with respect to its parameter (μ,ν), the pair of Beltrami differentials and which uniquely extends the Laplacians of hyperbolic metrics. Because of holomorphy of this family, the differential operators $\Delta_{\mu,\nu}$ cannot be self-adjoint off the diagonal $\{\mu = \nu\}$. These operators $\Delta_{\mu,\nu}$ are new examples of non-self-adjoint elliptic second order differential operators with a natural geometric origin!

We fix a Riemann surface X_0 modeled on the compact surface S. A Beltrami differential μ on X_0 is a complex $(-1,1)$-form which in one (and hence all) local representations

$$\mu = \mu(z)\frac{d\overline{z}}{dz}$$

satisfies $\|\mu\|_\infty < 1$. The space $M(X_0)$ of smooth Beltrami differentials on X_0 is a contractible complex analytic manifold modeled on a Fréchet space. The diagonal

$$\{(\mu,\mu) \mid \mu \in M(X_0)\} \subset M(X_0) \times \overline{M(X_0)}$$

is a totally real submanifold. Let G be the Fuchsian group of X_0, i.e. $X_0 = \mathbb{H}/G$. Denote by M^G the set of Beltrami differentials on $\mathbb{H}$ which transform as

$$\mu(z) = \mu(g(z))\frac{\overline{\partial g}}{\partial g}$$

for all $g \in G$. Then $M(X_0)$ is identified with M^G.

By $\hat{\mu}$ we denote a Beltrami differential on the lower half plane $\overline{\mathbb{H}}$ defined by $\hat{\mu}(z) = \overline{\mu}(\overline{z})$. Denote by $\overline{\partial}_\mu$ the operator $\overline{\partial} - \mu\partial$, and by $\partial_{\overline{\mu}}$ the operator $\partial - \overline{\mu}\overline{\partial}$.

Given a pair (μ,ν) of Beltrami differentials on $\mathbb{H}$, denote by $f_{\mu,\nu} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the homeomorphism of $\hat{\mathbb{C}}$ which is the unique continuous normalized solution (i.e. fixing 0, 1 and ∞) of the Beltrami equation on $\mathbb{C}$,

$$\begin{cases} \overline{\partial}_\mu f_{\mu,\nu} &= 0, \qquad \operatorname{Im} z > 0, \\ \overline{\partial}_{\hat{\nu}} f_{\mu,\nu} &= 0, \qquad \operatorname{Im} z < 0. \end{cases}$$

Denote $f^\mu = f_{\mu,\mu}$. By Ahlfors and Bers [AB] we know that the function $f_{\mu,\nu}$ depends holomorphically on the parameter (μ,ν).

Definition 0.1. Given a pair of Beltrami differentials $(\mu,\nu) \in M^G \times \overline{M^G}$, let

$$\alpha_{\mu,\nu} = \frac{1}{(1-\mu\overline{\nu})\partial f_{\mu,\nu}}.$$

Define a second order differential operator $\Delta_{\mu,\nu}$ on functions on $\mathbb{H}$ by the formula

$$\Delta_{\mu,\nu} = (f_{\mu,\nu} - \overline{f_{\nu,\mu}})^2(\partial\overline{\partial})_{\mu,\nu}$$

where

$$(\partial\overline{\partial})_{\mu,\nu} = \alpha_{\mu,\nu}\overline{\alpha_{\nu,\mu}}\big(-\mu\,\partial^2 + (1+\mu\overline{\nu})\,\overline{\partial}\partial - \overline{\nu}\,\overline{\partial}^2 + (\overline{\partial}_\mu \log\alpha_{\mu,\nu})\,\partial + (\partial_{\overline{\nu}}\log\overline{\alpha_{\nu,\mu}})\,\overline{\partial}\big).$$

The family $\{\Delta_{\mu,\nu}\}$ is the unique holomorphic extension of the Laplacian Δ in the sense that $\Delta_{\mu,\mu} = (f^\mu)^*\Delta$. The principal symbol of $\Delta_{\mu,\nu}$ in complex coordinates (z,ζ) on the cotangent bundle $T^*\mathbb{H}$, where $\sigma(\overline{\partial}) = i\zeta$, equals

$$\sigma_2(\Delta_{\mu,\nu})(\zeta) = -(f_{\mu,\nu} - \overline{f_{\nu,\mu}})^2\,\alpha_{\mu,\nu}\overline{\alpha_{\nu,\mu}}(\zeta - \mu\overline{\zeta})(\overline{\zeta} - \overline{\nu}\zeta).$$

By the identity $\overline{f_{\nu,\mu}(z)} = f_{\mu,\nu}(\overline{z})$, it is easy to see that the differential operator $\Delta_{\mu,\nu}$ is elliptic for any pair of Beltrami differentials (μ, ν). We can also show that $\Delta_{\mu,\nu}$ is invariant under the group G. So we have

Theorem 0.2. *There exists unique family of elliptic second order differential operators $\Delta_{\mu,\nu}$ on S parametrized by $(\mu,\nu) \in M(X_0) \times \overline{M(X_0)}$, with the following properties:*

1. *$\Delta_{\mu,\nu}$ depends holomorphically on (μ, ν);*
2. *the lift of $\Delta_{\mu,\mu}$ to $\mathbb{H}$ is the pull-back of the Laplacian $\Delta_{\mathbb{H}}$ by the quasiconformal mapping $f^\mu : \mathbb{H} \to \mathbb{H}$, i.e., $\Delta_{\mu,\mu}$ is the Laplacian of the hyperbolic metric on S induced by the pullback hyperbolic metric on $\mathbb{H}$ by the map f^μ.*

0.3. **Determinant of $\Delta_{\mu,\nu}$ and its holomorphy.** To define and study the determinant of the non-self-adjoint elliptic operator $\Delta_{\mu,\nu}$, we apply the method of using complex powers of elliptic operators developed by Seeley [Se1], [Se2]. (See [Sh] and [KV].) We need some condition on (μ, ν) to control the behavior of the spectrum of $\Delta_{\mu,\nu}$ to have a contour integral defining the complex power.

Given $0 < k < 1$ and $E > 0$, we introduce the space of Beltrami differentials

$$M_{k,E}(X_0) = \{\mu \in M(X_0) \mid \|\mu\|_\infty < k \ \text{ and } \ \|\mu\|_{C^2(X_0)} < E\}$$

where the C^2-norm $\|\cdot\|_{C^2(X_0)}$ is defined by the hyperbolic metric on X_0. We also define for each $\epsilon > 0$, an open subset

$$N_\epsilon = \{(\mu,\nu) \mid \mu,\nu \in M_{k,E}(X_0) \text{ and } \|\mu-\nu\|_{C^2(Q)} < \epsilon\} \subset M(X_0) \times \overline{M(X_0)}.$$

Theorem 0.3. *Given $0 < k < 1$ and $E > 0$, there exists a constant $\epsilon > 0$ such that on N_ϵ the determinant $\det'(\Delta_{\mu,\nu})$ is defined, and depends holomorphically on (μ, ν).*

Proof (an outline). We first establish several estimates for the normalized quasiconformal homeomorphism w^μ of $\mathbb{C}$, most importantly, the following pointwise lower bound of the first derivative.

Lemma 0.4. *Let $\Omega_1 \subset\subset \Omega \subset\subset \mathbb{C}$. If $\|\mu\|_{L^\infty} \le k < 1$, then*

$$\inf_{\Omega_1} |\partial w^\mu| \ge C e^{-C\|\partial\mu\|_{L^p(\Omega)}}$$

for $p = p(k) > 2$.

Fix $0 < \theta_0 < \pi$, then using the above lemma we can show

1. there exists $\epsilon > 0$ such that if $(\mu,\nu) \in N_\epsilon$, then
$$|\arg(\sigma_2(\Delta_{\mu,\nu}))| < \theta_0;$$
2. there exists a constant $C > 0$ such that for every μ, $\nu \in M_{k,E}(X_0)$ and for any nonzero eigenvalue λ of $\Delta_{\mu,\nu}$ on X_0,
$$|\lambda| \ge C - O(\|\mu-\nu\|_{C^2(X_0)}).$$

For the rest of proof denote $\Delta_{\mu,\nu}$ by A and assume that (μ,ν) belongs to where $\epsilon > 0$ will be determined in the following.

By (1), we know that for sufficiently small ϵ the principal symbol $\sigma_2(A)(x,\zeta)$ does not take values in the closed conical sector

$$\Lambda = \{\lambda : \theta_0 \leq \arg\lambda \leq 2\pi - \theta_0\}$$

in the spectral plane $\mathbb{C}$ for any $(x,\zeta) \in T^*S \setminus S$. This condition ensures that $\mathrm{Spec}(A) \cap \Lambda$ is finite, so there is a closed sector $\Lambda_0 \subset \Lambda$ which has only zero spectrum inside. By (2), we see that for sufficiently small $\epsilon > 0$, there is $\rho > 0$ such that

$$\mathrm{Spec}(A) \cap \{z \mid |z| < \rho\} \subset \{0\}.$$

Given $\exp(i\theta) \in \Lambda_0$, let $\Gamma_{(\theta)}$ be the contour $\Gamma_{1,\theta}(\rho) \cup \Gamma_{0,\theta}(\rho) \cup \Gamma_{2,\theta}(\rho)$, where

$$\begin{aligned}
\Gamma_{1,\theta}(\rho) &= \{x \exp(i\theta) \mid x \geq \rho\},\\
\Gamma_{0,\theta}(\rho) &= \{\rho \exp(i\phi) \mid \theta > \phi > \theta - \pi\},\\
\Gamma_{1,\theta}(\rho) &= \{x \exp(i(\theta - \pi)) \mid \rho \leq x\}.
\end{aligned}$$

Denote by R_λ the resolvent $(A - \lambda I)^{-1}$. Then for $\mathrm{Re}\, s < 0$, define

$$(A_s)_{(\theta)} = \frac{i}{2\pi} \int_{\Gamma_{(\theta)}} \lambda^s \, R_\lambda \, d\lambda.$$

By the symbol calculus of [Sh], A_s is trace class for $\mathrm{Re}\, s < -1$. In the following, we omit θ from the notation for $(A_s)_{(\theta)}$ and $\Gamma_{(\theta)}$.

For $s \in \mathbb{C}$, define the modified complex power $A^{s,o}$ of A by

$$A^{s,o} = A^k A_{s-k}$$

where k is an integer chosen so that $\mathrm{Re}\, s - k < 0$. The definition does not depend on the choice of such k. Following the arguments in [Sh] (pp. 94–106), we may show that the kernel $A^{-s,o}(x,y)\, dy$ of $A^{-s,o}$ can be meromorphically extended to all of $\mathbb{C}$, with simple poles contained in the set

$$\left\{ \frac{2-j}{2} \;\middle|\; j \geq 0 \right\} \setminus \{-j \mid j \geq 0\}.$$

It follows that the meromorphic function

$$\mathrm{Tr}(A^{-s,o}) = \int_M A^{-s,o}(x,x)\, dx$$

is regular at $s = 0$, and we can define

$$\det{}'(A) = \exp(-\partial_s|_{s=0} \mathrm{Tr}(A^{-s,o})).$$

As remarked by Kontsevich and Vishik in [KV], a change in the choice of contour Γ_θ changes $\partial_s|_{s=0} \mathrm{Tr}\, A^{-s,o}$ by an element of $2\pi i \mathbb{Z}$. After taking the exponential, the determinant $\det{}'(A)$ is well-defined.

For the holomorphy of $\det{}'(\Delta_{\mu,\nu})$ we use the following well-known variation formula for the determinant, which can be proved by symbol calculus of the kernel of complex powers.

$$d \log \det{}'(A) = \partial_s|_{s=0} \mathrm{Tr}(s A^{-s-1,o}\, dA).$$

In order to argue from this variation formula that $\det'(\Delta_{\mu,\nu})$ is holomorphic with respect to μ and ν, we must clarify one subtle point: the contour Γ must be chosen so that the spectrum of the operator $\Delta_{\mu,\nu}$ does not cross it as we perform the differentiation.ory In fact, we may choose for each $(\mu,\nu) \in N_\epsilon$ a contour Γ in such a way that the only eigenvalue of Δ_{μ_s,ν_t} inside Γ is zero, for any small variation (μ_s,ν_t) of (μ,ν) in N_ϵ. Since the determinant is independent of the choice of the contour, the holomorphy follows. $\square$

Denote by $\widetilde{\det'(\Delta)}$ the holomorphic extension of $\det'(\Delta)$ to $QF(S)$ obtained in Theorem 0.1. We have the principal fiber bundle

$$\begin{array}{ccc} \mathit{Diff}_0(S) & \longrightarrow & M(X_0) \\ & & \big\downarrow \pi \\ & & \mathit{Teich}(S), \end{array}$$

where the projection π is known to be holomorphic (see [EE]). This gives rise to the principal fiber bundle

$$\begin{array}{ccc} \mathit{Diff}_0(S) \times \mathit{Diff}_0(S) & \longrightarrow & M(X_0) \times \overline{M(X_0)} \\ & & \big\downarrow \pi\times\bar{\pi} \\ & & QF(S). \end{array}$$

The lift $(\pi \times \bar{\pi})^* \widetilde{\det'(\Delta)}$ is holomorphic on $M(X_0) \times \overline{M(X_0)}$. We know by Theorem 0.2 (2) that

$$\det'(\Delta_{\mu,\mu}) = (\pi \times \bar{\pi})^* \widetilde{\det'(\Delta)}(\mu,\mu),$$

and by Theorem 0.3 that the determinant $\det'(\Delta_{\mu,\nu})$ is defined and holomorphic on some open neighborhood N of the diagonal in $M(X_0) \times \overline{M(X_0)}$. Therefore, by analytic continuation, we have the equality

$$\det'(\Delta_{\mu,\nu}) = (\pi \times \bar{\pi})^* \widetilde{\det'(\Delta)}(\mu,\nu) \quad \text{for } (\mu,\nu) \in N,$$

and we may regard the holomorphic function $(\pi \times \bar{\pi})^* \widetilde{\det'(\Delta)}$ as the determinant of $\Delta_{\mu,\nu}$ even for those (μ,ν) to which Theorem 0.3 does not apply, that is, all of $M(X_0) \times \overline{M(X_0)}$.

Acknowledgment. The results of this paper are from the author's thesis for Northwestern University. He deeply thanks his thesis advisor Ezra Getzler for guidance and support. He is also grateful to Curtis McMullen, Peter Sarnak, András Vasy, and Jared Wunsch for helpful discussions or comments. He thanks anonymous referees for valuable remarks and suggestions.

References

[AB] L. V. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics.* Ann. of Math. **72** (1960), 385–404.

[BK] A. A. Belavin and V. G. Knizhnik, *Algebraic geometry and the geometry of quantum strings.* Phys. Lett. B **168**, no. 3, 201–206.

[EE] C. J. Earle and J. Eells, *A fiber bundle description of Teichmüller theory.* J. Diff. Geom. **3** (1969), 19–43.

[Fa] J. Fay, *Analytic torsion and prym differentials.* In: Riemann surfaces and related topics. Proceedings of the 1978 Stony Brook Conference, pp. 107–122, Ann. of Math. Stud., **97**, Princeton Univ. Press, Princeton, N.J., 1981.

[Ki] Y.-H. Kim, *Holomorphic extensions of Laplacians and their determinants.* `math.CV/0505530`.

[KV] M. Kontsevich and S. Vishik, *Determinants of elliptic pseudo-differential operators.* `hep-th/9404046`.

[Mc] C. T. McMullen, *The moduli space of Riemann surfaces is Kähler hyperbolic.* Ann. of Math. **151** (2000), 327–357.

[OPS1] B. Osgood, R. Phillips, and P. Sarnak, *Extremals of determinants of Laplacians.* J. Funct. Anal. **80** (1988), 148–211.

[OPS2] B. Osgood, R. Phillips, and P. Sarnak, *Compact isospectral sets of surfaces.* J. Funct. Anal. **80** (1988), 212–234.

[Po] A. M. Polyakov, *Quantum geometry of bosonic strings.* Phys. Lett. B **103** (1981), no. 3, 207–210.

[Pl] I. Platis, *Complex symplectic geometry of quasi-fuchsian space.* Geometriae Dedicata **87** (2001), 17–34.

[RS1] D. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds.* Adv. Math. **7** (1971), 145–210.

[RS2] D. Ray and I. M. Singer, *Analytic torsion for complex manifolds.* Ann. Math. **98** (1973), 154–177.

[Sa1] P. Sarnak, "Some applications of modular forms." Cambridge Tracts in Math., vol 99, Cambridge University Press, Cambridge, 1990.

[Sa2] P. Sarnak, *Extremal geometries.* In "Extremal Riemann surfaces (San Francisco, CA, 1995)." Contemp. Math. **201**, Amer. Math. Soc., Providence, RI, 1997, pp. 1–7.

[Se1] R. T. Seeley, *Complex powers of an elliptic operator.* In "Singular integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966)" Amer. Math. Soc., Providence, R.I., pp. 288–307.

[Se2] R. T. Seeley, *The Resolvent of an elliptic boundary problem.* Amer. J. Math. **91** (1969), 889–920.

[Sh] M. Shubin, "Pseudodifferential operators and spectral theory." Second edition. Springer-Verlag, Berlin, 2001.

[TM] L. P. Teo and A. McIntyre, *Holomorphic factorization of determinants and quasifuchsian groups*, preprint. near botto

[W1] S. Wolpert, *Chern forms and the Riemann tensor for the moduli space of curves.* Invent. Math. **85** (1986), 119–145.

[W2] S. Wolpert, *Asymptotics of the spectrum and the Selberg zeta function of Riemann surfaces.* Comm. Math. Phys. **112** (1987), 283–315.

[ZT] P.G. Zograf and L.A. Takhtadzhyan, *A local index theorem for families of $\overline{\partial}$-operators on Riemann surfaces.* Russian Math. Surveys **42** no. 6 (1987), 169–190.

Department of Mathematics, Northwestern University, Evanston, IL 60208, USA

Current address The Fields Institute for Research in Mathematical Sciences, 222 College Street, Toronto, Ontario, M5T 3J1 Canada

E-mail address: `ykim@fields.utoronto.ca`