Curvature and the continuity of optimal transportation maps

Young-Heon Kim and Robert J. McCann

Department of Mathematics, University of Toronto

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Monge-Kantorovich Problem

Mass Transportation

- $M^n, \bar{M}^n$ bounded domains in $\mathbb{R}^n$;
  compact manifolds (e.g. S^n, $T^n = \mathbb{R}^n/\Gamma$)
- $\rho \in Prob(M), \bar{\rho} \in Prob(\bar{M})$, Borel probability measures

"**weighted multi-valued maps γ**"

$$\Gamma(\rho, \bar{\rho}) := \{\gamma \in Prob(M \times \bar{M}) \mid [\pi_M]_\# \gamma = \rho,\ [\pi_{\bar{M}}]_\# \gamma = \bar{\rho}\}$$

Push-forward: $f : X \to Y$

$$f_\# \gamma(U) = \gamma(f^{-1}(U)) \ \ \forall U \subset Y$$

- $c : M \times \bar{M} \to \mathbb{R} \cup \infty$ **transportation cost function**
  lower-semi continuous
- **Transport Cost**
  for the weighted multi-valued map $\gamma \in \Gamma(\rho, \bar{\rho})$

$$\mathcal{C}(\gamma) := \int_{M \times \bar{M}} c(x, \bar{x}) d\gamma(x, \bar{x}).$$

Minimize ?

$$\inf_{\gamma \in \Gamma(\rho, \bar{\rho})} \mathcal{C}(\gamma) \text{ ?}$$

Well-known [Kantorovitch 1942]:

$\exists$ optimal γ. **(Kantorovitch solution)**

- [Brenier 87, McCann-Gangbo 95, Caffarelli 96, McCann 01, ...]
  $M, \bar{M}$ a smooth manifold, $\rho \ll d\,\text{vol}$
  Twisted **(A1)** cost c (e.g. $c(x,y) = |x-y|^2$ on $\mathbb{R}^n$)
  THEN

$$\exists\,!\ \text{optimal}\ \gamma$$

and **(Monge-Kantorovich map)**

$$\exists\,!\ \text{Borel measurable}\ F : M \longrightarrow \bar{M}$$

such that

$1) F_{\#}\rho = \bar{\rho}$,

2) $\text{spt}\gamma\ \subset_{a.e}$ $\mathrm{Graph}(F) := \{(x,\bar{x}) \in M \times \bar{M} \mid \bar{x} = F(x)\}$

Question (**Regularity of F ?**)

The optimal (Monge-Kantorovitch) map $F : M \to \bar{M}$

- **? Continuous** *if*
  $0 \le \rho \in L^1(M) \cap L^\infty(M), 0 < \epsilon \le \bar{\rho} \in L^1(\bar{M}) \cap L^\infty(M)$ *?*
- **? Smooth** *if*
  $0 \le \rho \in C^\infty(M) \cap L^\infty(M), 0 < \bar{\rho} \in C^\infty(\bar{M}) \cap L^\infty(M)$*?*

Necessity:

- Condition on Domain, e.g. $\bar{M}$ is connected, and more.....
- Condition on the cost function c

Example (Loeper 07p)

$c = dist^2$ on a saddle surface ($K < 0$) $\Rightarrow$ continuity of F fails.

Regularity of F (M & $\bar{M} \subset \mathbb{R}^n$, $\log \bar{\rho} \in L^\infty(\bar{M})$)

- $c(x, \bar{x}) = |x - \bar{x}|^2/2$,
  - Delanoë (91) for $n = 2$
  - Caffarelli (92)(96) using geometric ideas
  - Urbas, (97) continuity method in PDE.

  Under $\bar{M} \subset \mathbb{R}^n$ is convex.
- For general costs c,
  - Ma, Trudinger & Wang (05)(07p)

    (A0) (A1) (A2) (A3w) on c

    $+$

    geometric conditions ("c-convexity") on M, $\bar{M}$

    $\Rightarrow$ continuity and smoothness of optimal maps

  - Loeper (07p):
    (A3w) is necessary for the continuity of F,
  - ([Loeper 07p]
    \+ a key technical lemma [K-M] ([Trudinger & Wang 07p-2])
    - (A3s) $\Rightarrow$ Hölder continuity of F even for very rough ρ (ρ not absolutely continuous w.r.t *Lebesgue*).

Regularity of F

$M, \bar{M} = S^n$

- (Reflector Antenna Problem, Oliker, Wang, ...)
  $c(x, y) = -\log|x - y|$, for $S^n \subset \mathbb{R}^{n+1}$
  Wang (96, 04) Caffarelli, Gutierrez & Huang (07) Loeper (07p) $+$ a key technical lemma [K-M] ([Trudinger & Wang 07p-2])
  - Hölder continuity of F when $\rho, \bar{\rho} \in L^1$
- (Distance squared) $c = dist^2$
  [Loeper 07p]
  $+$ either a key technical lemma [K-M] ([Trudinger & Wang 07p-2])
  {Or [Delanoë 04] & [Ma, Trudinger, & Wang 05] }
  - Hölder continuity of F with rough mass ρ

Note: c is not differentiable on $S^n \times S^n$

When do we have continuity and smoothness results? Conditions (A0) (A1) (A2) (A3w) (A3s)

Originally for

$$N = M \times \bar{M} \subset \mathbb{R}^n \times \mathbb{R}^n$$

[Ma, Trudinger, Wang05][Trudinger, Wang07p]

A0 (Smoothness of c)

A0 $c \in C^4(N)$

$N \subset M \times \bar{M}$ is THE LARGEST open subset where c is SMOOTH.
$\bar{N}(x) = \{\bar{z} \in \bar{M} \mid (x, \bar{z}) \in N\}$ "accessible from x"
$N(\bar{x}) = \{z \in M \mid (z, \bar{x}) \in N\}$ "accessible from $\bar{x}$"

Example

- $c(x, y) = d^2(x, y)$ the Riemannian distance squared cost on $S^n \times S^n$
  which is NOT differentiable on $\{(x, \hat{x})\}$, $\hat{x}$ the antipodal of x.
  $N = S^n \times S^n \setminus \{(x, \hat{x})\}$

A1 (Twist condition), A2 (Non-degeneracy)

$$D = \partial_{x^i}\, dx^i \qquad \text{differential on } M$$

$$\bar{D} = \partial_{\bar{x}^j}\, d\bar{x}^j \qquad \text{differential on } \bar{M}$$

one-form $\quad dc = Dc \oplus \bar{D}c.$

A1 **(Twist condition)**
$\bar{y} \in \bar{N}(x) \mapsto -Dc(x, \bar{y}) \in T^*_x M,$
$y \in N(\bar{x}) \mapsto -\bar{D}c(y, \bar{x}) \in T^*_{\bar{x}} \bar{M}$
both injective $\forall (x, \bar{x}) \in N$

A2 **(Non-degeneracy)**
the linear map
$D\bar{D}c : T_{\bar{x}}\bar{M} \longrightarrow T^*_x M$
injective.
i.e.
$D\bar{D}c$ **non-degenerate** 2-form on N

A3?

[Ma, Trudinger, Wang05][Trudinger, Wang 07p]

For

$$N = M \times \bar{M} \subset \mathbb{R}^n \times \mathbb{R}^n$$

Definition

(A3w)

$$c^{\bar{k}a} c^{\bar{l}b} [c_{ij\bar{k}\bar{l}} + c_{ij\bar{n}} c^{\bar{n}m} c_{m\bar{k}\bar{l}}] \xi^i \xi^j \eta^a \eta^b \geq 0$$

for all $\xi \perp \eta$

Definition

(A3s)

$$c^{\bar{k}a} c^{\bar{l}b} [c_{ij\bar{k}\bar{l}} + c_{ij\bar{n}} c^{\bar{n}m} c_{m\bar{k}\bar{l}}] \xi^i \xi^j \eta^a \eta^b > \delta |\xi|^2 |\eta|^2$$

for all $\xi \perp \eta$

Example ((A0)(A1)(A2)(A3w))

$c(x,y) = \frac{1}{2}|x-y|^2$ on $\mathbb{R}^n \times \mathbb{R}^n$, ...

Example ((A0)(A1)(A2)(A3s))

- $c(x,y) = -\log|x-y|$ on $\mathbf{R}^n \times \mathbf{R}^n \setminus \Delta$, $\Delta = \{(x,x)\}$.
- $c(x,y) = -\log|x-y|$ on $S^n \times S^n \setminus \Delta \subset \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ "Reflector Antenna Problem"
- $c(x,y) = \sqrt{1+|x-y|^2}$ on $\mathbb{R}^n$, ...
- [Loeper 07p] $c(x,y) = dist^2(x,y)$ on $S^n \times S^n \setminus \{(x,\hat{x})\}$

A3 and Pseudo-Riemannian metric h

Definition ([K-M])

(Pseudo-Riemannian metric h) $N \subset M \times \bar{M}$
$T_{(x,\bar{x})}N = T_x M \oplus T_{\bar{x}}\bar{M}$ and $T^*_{(x,\bar{x})}N = T^*_x M \oplus T^*_{\bar{x}}\bar{M}$.
Thus

$$h := \begin{pmatrix} 0 & -D\bar{D}c \\ -(D\bar{D}c)^\dagger & 0 \end{pmatrix} \tag{1.1}$$

symmetric & bilinear form on $T_{(x,\bar{x})}N$.

- **non-degenerate** by **A2**
- **NOT positive-definite**

$\Rightarrow$ a **pseudo-Riemannian**, of (n, n) type, metric on N.
eigenvectors $p \oplus \bar{p}$ and $(-p) \oplus \bar{p}$.

The Riemann curvature tensor R_{ijkl} of h
("Second derivatives of h" ="Fourth derivatives of c"
Recall $c \in C^4(N)$)
the sectional curvature

$$\sec_{(x,\bar{x})}(P \wedge Q) = \sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{k=1}^{2n}\sum_{\ell=1}^{2n} R_{ijkl} P^i Q^j P^k Q^\ell \quad (1.2)$$

Definition ([K-M])

(Cross-curvature)

- **(A3w)** $\forall (x, \bar{x}) \in N$

$$\sec_{(x,\bar{x})}((p \oplus 0) \wedge (0 \oplus \bar{p})) \geq 0$$

$\forall$ **null vectors** $p \oplus \bar{p} \in T_{(x,\bar{x})}N$ (i.e. $h(p \oplus \bar{p}, p \oplus \bar{p}) = 0$).

- **(A3s)**

$$\text{(A3w)} + \{\textit{equality in } (A3w) \Rightarrow p = 0 \textit{ or } \bar{p} = 0\}$$

Cross-curvature and Riemannian curvature

Riemannian manifold $(M = \bar{M}, g)$. $c(x, \bar{x}) = dist^2(x, \bar{x})/2$
$\dashrightarrow$ a metric tensor h on $N = M \times M \setminus \text{cut locus}$.

- $\sqrt{2}M \cong \Delta := \{(x, x) \mid x \in M\}$, $h(p \oplus \bar{p}, p \oplus \bar{p}) = 2g(p, \bar{p})$
- $\Delta \hookrightarrow N$ totally geodesic (by the symmetry $c(x, \bar{x}) = c(\bar{x}, x)$)
- h-nullity of $p \oplus \bar{p} \in T_{(x,x)}N \Leftrightarrow g$-orthogonality of p with $\bar{p}$.
-

$$\sec^{(N,h)}_{(x,x)}(p \oplus 0) \wedge (0 \oplus \bar{p}) = \frac{2}{3} \sec^{(M,g)}_{x} p \wedge \bar{p}$$

thus
negative curvature at any point on (M, g)
$\Rightarrow$ a violation of cross-curvature non-negativity **(A3w)** on (N, h).
$\Rightarrow$ Loeper's counter example to continuity of optimal maps.

Example

$c(x,y) = dist^2(x,y)$ on a Riemannian manifold (e.g. $M = \bar{M} = S^n$)
$t \in \mathbf{R} \to (x, \bar{x}(t)) \in M \times \bar{M}$ a h-geodesic:
$\Longleftrightarrow$
$\bar{x}(t) = \exp_x((1-t)p + tq)$, for some $p, q \in T_xM$.

Is pseudo-Riemannian formalism BETTER ?

- No other structures needed than the cost c
  - Coordinate free (covariant):
  Continuity (regularity) is coordinate free notion
  i.e. doesn't change under diffeomorphic change
  Any notion explaining regularity phenomena should be coordinate free, like **cross-curvature** .
- "convexities" in this h-geometry
  gives
  the necessary conditions on domains M, $\bar{M}$
  for continuity of F
- Geometric formulation and proof of a key technical result ("DASM") for the continuity of F.

Convexity in h-geometry

$N \subset M \times \bar{M}$
$\bar{N}(x) = \{\bar{z} \in \bar{M} \mid (x, \bar{z}) \in N\}$ "accessible from x"
$N(\bar{x}) = \{z \in M \mid (z, \bar{x}) \in N\}$ "accessible from $\bar{x}$"

- N is ***forward convex*** if $\{x\} \times \bar{N}(x)$ is h-geodesically convex $\forall x \in M$.
- N is ***backward convex*** if $N(\bar{x}) \times \{\bar{x}\}$ is h-geodesically convex $\forall \bar{x} \in \bar{M}$.
- N is ***bi-convex*** if N is forward and backward convex.

Example

$M = \bar{M} = \mathbf{R}^n \cup \{\infty\}$.
$c(x, y) = -\log|x - y|$ on $N = M \times M \setminus \Delta$.
N is bi-convex

A basic lemma in h-geometry

Lemma ([K-M])

h the pseudo-Riemannian metric on $N \subset M \times \bar{M}$.
$\sigma(s) = (x(s), \bar{x}(0))$,
$\tau(t) = (x(0), \bar{x}(t))$;
$\sigma(0) = \tau(0) = (x(0), \bar{x}(0))$.
*Suppose $s \in [-1, 1] \longrightarrow \sigma(s) \in N$ $\quad h$-**geodesic**.*
THEN

$$-\frac{\partial^4}{\partial s^2 \partial t^2}\bigg|_{s=0=t} c(x(s), \bar{x}(t)) = \mathrm{sec}_{(x(0),\bar{x}(0))} \frac{d\sigma}{ds} \wedge \frac{d\tau}{dt}.$$

A Geometric Result in h-geometry **("DASM": Double Mountain Above Sliding Mountain)**

Theorem ([K-M])

$c \in C^4(N) \cap C(M \times \bar{M})$.
The pseudo-Riem. metric h has **non-negative cross-curvature** *on* $N \subset M \times \bar{M}$. *(A0, A1, A2, A3w).*
N *is* **backward-convex**.
$t \in [0,1] \longrightarrow (x, \bar{x}(t)) \in N$ *be a* h-**geodesic** *in* N.
Suppose $\bigcap_{0<t<1} N(\bar{x}(t))$ *dense in* M. *(e.g.* $N = M \times \bar{M}$*)*
THEN
The function $f_t(\cdot) = -c(\cdot, \bar{x}(t)) + c(x, \bar{x}(t))$ *on* M *satisfies*

$$\textbf{"DASM"} \quad f_t \leq \max\{f_0, f_1\} \ \text{ on } 0 \leq t \leq 1.$$

Proof of DASM

For simplicity suppose (A3s): Cross-Curvature > 0.

Let $x \neq y \in \bigcap_{0<t<1} N(\bar{x}(t))$.

Claim: Given t, $\frac{\partial}{\partial t} f_t(y) = 0 \Rightarrow \frac{\partial^2}{\partial t^2} f_t(y) > 0$.

- $f_\tau(x) = 0, \forall \tau$. Thus $\frac{\partial}{\partial t} f_t(x) = \frac{\partial^2}{\partial t^2} f_t(x) = 0$.
- By backward-convexity of N
  $\exists$ ***h*-geodesic** $s \in [0,1] \longrightarrow (x(s), \bar{x}(t))$ in $N(\bar{x}(t)) \times \{\bar{x}(t)\}$,
  - $x(0) = x$ and $x(1) = y$;
  - $x(s) \in \{\frac{\partial}{\partial t} f_t = 0\}$, $\forall s \in [0,1]$;
  - $\frac{d}{dt}\bar{x}(t) \oplus \frac{d}{ds}x(s)$ is **null** $\forall s \in [0,1]$
-

$$\frac{\partial^2}{\partial s^2}\Big[\frac{\partial^2}{\partial t^2} f_t\Big](x(s)) = \sec_{(x(s),\bar{x}(t))}\Big(\frac{d}{ds}x(s) \oplus 0\Big) \wedge \Big(0 \oplus \frac{d}{dt}\bar{x}(t)\Big)$$
$$> 0 \quad \text{(by A3s)},$$

- $\frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial^2}{\partial t^2} f_t(x(s)) = 0$ ($\Leftarrow$ h-geodesic equation of $(x, \bar{x}(t))$)

$\Rightarrow\Rightarrow$ $\frac{\partial^2}{\partial t^2} f_t(y) > 0$.

Economic interpretation of "DASM"

- **Supply**: ρ **Demand**: $\bar{\rho} = \epsilon\delta_{\bar{y}} + (1-\epsilon)\delta_{\bar{z}}$
  $c(x, \bar{x})$: the cost of transportation
  $\lambda = Price(\bar{y}) - Price(\bar{z})$
- **Economic equilibrium**: $\exists\ \lambda \in \mathbf{R}$ such that

$$u(x) = \max\{\lambda - c(x,\bar{y}), -c(x,\bar{z})\}$$
$$\text{yields} \quad \epsilon = \rho\big[\{x \in M \mid u(x) = \lambda - c(x,\bar{z})\}\big]$$

- **valley of indifference**: $V = \{x \in M \mid c(x,\bar{y}) - c(x,\bar{z}) = \lambda\}$
- **"DASM"** $x_0 \in V \Rightarrow x_0$ indifferent to $\bar{x}(t)$, $\bar{y} = \bar{x}(0)$ to $\bar{z} = \bar{x}(1)$; i.e. $\exists\ \lambda(t)$, such that

$$u(x) \geq \max_{0 \leq t \leq 1} \lambda(t) - c(x,\bar{x}(t)) \quad \text{with } "=" \text{at } \ x_0$$
$$\lambda(t) = c(x_0,\bar{x}(t)) - c(x_0,\bar{z});$$
$$t \in [0,1] \longrightarrow (x_0,\bar{x}(t)) \text{ is a } h\text{-geodesic}$$

"DASM"

- **Remark:**
  "DASM" fails
  $\Rightarrow$ continuity of optimal map F fails. [Loeper, 07p]
- **Consequences of "DASM"**
  - Makes Loeper's proof of Hölder continuity (with rough ρ) self-contained
    - Riemannian distance squared cost
      $c = dist^2$ on $M = \bar{M} = S^n$.
    - "Reflector Antenna Problem"
  - relax geometric hypotheses on $M, \bar{M}$ and the cost c which previous authors required
  - Hölder continuity result for (some) other cost functions on manifolds. (e.g. $c(x, y) = \sqrt{1 + dist^2(x, y)}$ on $\mathbf{R}^n/\mathbf{Z}^n$.)
  - First step toward: general Regularity theory of optimal transportation on manifolds

Thank You Very Much