

COUNTEREXAMPLES TO CONTINUITY OF OPTIMAL TRANSPORTATION ON POSITIVELY CURVED RIEMANNIAN MANIFOLDS

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ABSTRACT. Counterexamples to continuity of optimal transportation on Riemannian manifolds with everywhere positive sectional curvature are provided. These examples show that the condition **A3w** of Ma, Trudinger, & Wang is not guaranteed by positivity of sectional curvature.

1. INTRODUCTION

This paper addresses a question (see Question 1.1) in both optimal transportation theory and Riemannian geometry. The question is explained in the following. For general notions we refer to the books by Villani [V1] [V2] for optimal transport theory and the book by Cheeger and Ebin [CE] for Riemannian geometry.

In optimal transportation, one considers two measure distributions $\rho, \bar{\rho}$ — with the same total measure — on domains M, \bar{M} , respectively, and one seeks for a minimizing (measurable) map $F : M \rightarrow \bar{M}$ for moving ρ to $\bar{\rho}$ while it costs certain amount to move each unit mass at one location to another: this cost is given as a real valued function $c = c(x, \bar{x})$ on the product $M \times \bar{M}$.

The case $c(\cdot, \cdot) = \text{dist}^2(\cdot, \cdot)/2$ for Riemannian distance dist on a Riemannian manifold $M = \bar{M}$ has been of great interest among researchers and an existence and uniqueness theory of optimal maps F has been known for this case by the works of Brenier [B] for Euclidean spaces and McCann [Mc] for general Riemannian manifolds. Note that the distance squared cost c (when differentiable) satisfies $\nabla_x c(x, \bar{x}) = (\exp_x)^{-1}(\bar{x})$, and thus it can be regarded as the canonical cost function for a Riemannian manifold — when we say about a Riemannian manifold in this paper we always mean the manifold together with its distance squared cost.

The present work concerns the regularity of optimal transportation maps for Riemannian distance squared costs. A key notion is the so-called *A3 weak condition* denoted as **A3w** (see Definition 2.2). Ma, Trudinger, and Wang [MTW][TW1][TW2] have introduced and used this notion to develop a regularity theory of optimal transportation maps for general cost functions extending the results of Delanoë [D1], Caffarelli [Ca1][Ca2], and Urbas [U] for Euclidean distance squared costs. In fact, this **A3w** is a necessary condition for continuity of optimal transport maps

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as shown later by Loeper [Lo]: he showed that if **A3w** is violated then there exist smooth source and target measures $\rho, \bar{\rho}$ such that the optimal transportation map F is not even continuous. Moreover, for Riemannian distance squared costs, Loeper [Lo] has shown that to satisfy **A3w** the manifold should have nonnegative sectional curvature everywhere, and the standard round spheres S^n satisfy **A3w** (in fact a stronger condition so-called **A3s**). This has led him to understand **A3w** as a certain curvature condition, and to ask the following natural question.

Question 1.1. (nonnegative curvature \implies A3w ?) *Does every nonnegatively curved — the sectional curvature is nonnegative everywhere — Riemannian manifold satisfy A3w for its distance squared cost?*

As the main result of this paper, we answer Question 1.1 *negatively* by showing counterexamples.

Main Theorem 1.2. (nonnegative or positive curvature $\not\Rightarrow$ A3w) *For each dimension n , there are complete (compact or noncompact) n -dimensional Riemannian manifolds with everywhere positive (nonnegative) curvature which do not satisfy A3w.*

Proof. This result follows from Theorem 5.1, Corollary 5.2, and Remark 5.1: it is shown that some shallow, smooth convex cones — which are nonnegatively curved — do not satisfy **A3w**; by perturbation, positively curved examples are also obtained. \square

As far as the present author knows, the examples we construct in this paper are the first examples of nonnegatively or positively curved Riemannian manifolds where there are discontinuous optimal maps for smooth source and target measures $\rho, \bar{\rho}$. These examples confirm Trudinger’s suspicion [T] about Question 1.1.

Some perspectives on A3w and Main Theorem 1.2. Let’s first discuss **A3w** in some detail. In its original form as introduced by Ma, Trudinger, & Wang (to show regularity of Monge-Ampère type equations arising from optimal transportation theory), **A3w** has been mysterious to researchers. The first geometric interpretation of **A3w** is given by Loeper [Lo] (see Theorem 2.3). The present author and Robert McCann give another more conceptual geometric interpretation [KMc2] by introducing a pseudo-Riemannian metric, say h , on the product space $M \times \bar{M}$ of the source and target domains of the same dimension. This metric h is defined using the mixed second order partial derivatives of the cost function $c : M \times \bar{M} \rightarrow \mathbf{R}$ as the following non-degenerate¹ symmetric² bilinear form³ on

¹The non-degeneracy of h needs the non-degeneracy of $D\bar{D}c$, a condition called **A2** by Ma, Trudinger, & Wang [MTW] [TW1], and each Riemannian distance squared cost on $M = \bar{M}$ (when differentiable) satisfies **A2**.

² $D\bar{D}c$ is the adjoint of $\bar{D}Dc$.

³This bilinear form is of type (n, n) , i.e. it has the same number of positive and negative eigenvalues; $n = \dim M = \dim \bar{M}$.

$TM \oplus T\bar{M}$:

$$(1.1) \quad h := \begin{pmatrix} 0 & -\frac{1}{2}\bar{D}Dc \\ -\frac{1}{2}(D\bar{D}c) & 0 \end{pmatrix}$$

where D, \bar{D} denote the differentials of each M, \bar{M} , respectively. Then **A3w** is equivalent to the following nonnegativity condition for the curvature of this pseudo-Riemannian metric h : namely, for each $(x, \bar{x}) \in M \times \bar{M}$ and each tangent vector $p \oplus \bar{p} \in T_{(x, \bar{x})}M \times \bar{M} = T_x M \oplus T_{\bar{x}} \bar{M}$ at (x, \bar{x}) ,

$$(1.2) \quad R_h((p \oplus 0) \wedge (0 \oplus \bar{p}), (p \oplus 0) \wedge (0 \oplus \bar{p})) \geq 0 \text{ if } h(p \oplus \bar{p}, p \oplus \bar{p}) = 0,$$

where R_h denotes the curvature operator of h . The left-hand side of the inequality in (1.2) is called *cross-curvature*. Thus, **A3w** can be interpreted as nonnegativity condition for cross-curvature of null-planes in h -geometry⁴. For the Riemannian distance squared cost $c(x, \bar{x}) = \text{dist}^2(x, \bar{x})/2$ on a Riemannian manifold $M = \bar{M}$, M is totally geodesically embedded as the diagonal of $M \times M$ with respect to the pseudo-Riemannian metric h , and the cross-curvature in (1.2) along this diagonal coincides with $R_M(p \wedge \bar{p}, p \wedge \bar{p})$ where R_M denotes the curvature operator of M (see [KMc2] for details). This is another way to see Loeper's result [Lo] that **A3w** implies nonnegative sectional curvature.

The pseudo-Riemannian metric h and its curvature — though they are local in the product space $M \times \bar{M}$ — are global in nature with respect to the geometry of M and \bar{M} : i.e. for a Riemannian manifold $M = \bar{M}$, local information concerning h is equivalent to information about the global distance structure of M . Therefore **A3w** — the nonnegative cross-curvature condition for null-planes of h — is expected to be a stricter restriction than nonnegative sectional curvature condition of M . Main Theorem 1.2 confirms this. As a consequence, this makes the following question of Trudinger [T] much more interesting.

Question 1.3. *For a Riemannian manifold M with everywhere positive sectional curvature, let R_M denote the curvature operator. Does there exist certain $\epsilon > 0$ such that if*

$$\|\nabla \log \|R_M\|\| \leq \epsilon \text{ with an appropriate point-wise norm } \|\cdot\|,$$

*then M satisfies **A3w**?*

Note that so far the only known examples of **A3w** Riemannian manifolds⁵ are modulo C^∞ -perturbations — C^2 is maybe enough — the Euclidean space \mathbf{R}^n (without perturbation), the standard n -dimensional sphere S^n [Lo], and the Riemannian

⁴A Riemannian manifold is said to be *non-negatively cross-curved* if the inequality in (1.2) holds without the condition $h(p \oplus \bar{p}, p \oplus \bar{p}) = 0$.

⁵On the other hand, there are a lot of known examples of other types of cost functions satisfying **A3w** [MTW] [TW1].

manifolds obtained from these by Riemannian coverings as considered by Cordero-Erausquin [Co], Delanoë [D2], and Delanoë and Ge [DG], more generally by Riemannian submersions⁶ and products⁷ as shown by the present author and McCann [KMc3]⁸(see also [KMc2]). For all these unperturbed examples, $\|R_M\| = \text{const}$.

Organizational remarks. Although the notions and terminology in this paper have more general versions, they are specialized to Riemannian distance squared costs for the sake of expositional simplicity.

This paper is organized as follows. In Section 2 some preliminary notions and results are presented; Section 3 explains the key idea of the counterexamples we construct; Section 4 shows a Riemannian geometric result which is used in the main theorem; Section 5 is devoted to the construction of the counterexamples — the main theorem of this paper.

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2. PRELIMINARIES

In this section, some preliminary results, the definitions of **A3w** and other key notions are presented.

First, let's recall a 2-dimensional version of the famous theorem of Toponogov⁹ which is essentially used in the proof of main theorem (Theorem 5.1).

Theorem 2.1. (Toponogov's comparison theorem) *Let M be a complete 2-dimensional Riemannian manifold with sectional curvature $K_M \geq H$, and let M^H be the simply connected 2-dimensional space of constant curvature H . Let $\gamma_i : [0, 1] \rightarrow M$ and $\bar{\gamma}_i : [0, 1] \rightarrow M^H$, $i = 1, 2$, be minimal geodesic segments, i.e. they are unique geodesic segments connecting their end points. Suppose that $\gamma_1(0) = \gamma_2(0)$, $\bar{\gamma}_1(0) = \bar{\gamma}_2(0)$; $\angle(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) = \angle(\dot{\bar{\gamma}}_1(0), \dot{\bar{\gamma}}_2(0)) < \pi$, where \angle denotes the*

⁶For example, the complex projective space \mathbf{CP}^n with its Fubini-Study metric — the sectional curvature $1 \leq K_{\mathbf{CP}^n} \leq 4$.

⁷For example, $M_1 \times \cdots \times M_k$, where $M_i = \mathbf{R}^l, S^m$, or \mathbf{CP}^n for $i = 1, \dots, k$.

⁸In fact, we showed that (1) S^n with its standard round metric is non-negatively cross-curved; (2) Riemannian submersions of **A3w**/**A3s** (resp. non-negatively cross-curved) Riemannian manifolds always induce **A3w**/**A3s** (resp. non-negatively cross-curved) Riemannian manifolds; (3) for products, if each factors are non-negatively cross-curved, then the resulting manifolds are non-negatively cross-curved, thus **A3w** (but never **A3s**); (4) and moreover, if one of the factors is not non-negatively cross-curved then the product is not **A3w**. See [KMc2][KMc3] for details and generalizations.

⁹The full version can be found in [CE].

angle between tangent vectors. Assume $L[\gamma_i] = L[\bar{\gamma}_i]$, $i = 1, 2$, where L denotes arc-length. Then

$$(2.1) \quad \text{dist}(\gamma_1(1), \gamma_2(1)) \leq \text{dist}(\bar{\gamma}_1(1), \bar{\gamma}_2(1)),$$

where dist denotes the Riemannian distance. Moreover, if there exists a point z on $\gamma_1 \cup \gamma_2 \subset M$ such that $K_M(z) > H$, then the inequality (2.1) is strict.

The following key notions are specialized to Riemannian distance squared costs for the sake of expositional brevity: in fact, they have more general definitions [MTW] [TW1] [Lo] [KMc2][V2].

Definition 2.2. (c-segment, A3w, and local DASM) Let M be a complete Riemannian manifold and let c denote the Riemannian distance squared cost, i.e. $c(x, y) = \text{dist}^2(x, y)/2$ for $x, y \in M$.

- **(c-segment)**[MTW] A curve $t \in [0, 1] \rightarrow M$ is called a c -segment with respect to x , if $\bar{x}(t) = \exp_x(p + t\eta)$, for some $p, \eta \in T_x M$ and $c(x, \bar{x}(t)) = |p + t\eta|^2/2$.
- **(A3w)**[MTW][TW1] M is said to satisfy **A3w** if for any triple $(x, \bar{x}(t), \eta)$ of a point $x \in M$, a c -segment $t \in [0, 1] \rightarrow \bar{x}(t) = \exp(p + t\eta)$, $p, \eta \in T_x M$, and a tangent vector $\xi \in T_x M$ with $\eta \perp \xi$,

$$(2.2) \quad \frac{d^2}{dt^2} \Big|_{t=0} [-D_{xx}^2 c](x, \bar{x}(t)) \xi \xi \geq 0,$$

where D_{xx}^2 denotes the Riemannian Hessian with respect to the first argument of c .

Remark 2.1. Loeper [Lo] calls the left-hand-side of the inequality (2.2) *cost-sectional curvature* — he has shown it coincides with Riemannian sectional curvature when $x = \bar{x}(0)$.

- **(local DASM)**¹⁰[Lo] (c.f. [KMc2]) M is said to satisfy **local DASM** if for any $x \in M$ and any c -segment $t \in [0, 1] \rightarrow \bar{x}(t)$ with respect to x , there exists a neighborhood U of x such that the function $f_t(\cdot) = -c(\cdot, \bar{x}(t)) + c(x, \bar{x}(t))$ satisfies

$$(2.3) \quad f_t(y) \leq \max[f_0(y), f_1(y)], 0 \leq t \leq 1, \quad \forall y \in U.$$

The notion **local DASM** can be understood as a geometric interpretation of **A3w** because of the following theorem which is originally due to Loeper [Lo].

Theorem 2.3. (A3w \iff local DASM) Let M be a complete Riemannian manifold. M satisfies **A3w** if and only if M satisfies **local DASM**.

Proof. (\implies) This direction can be easily verified by the elementary and geometric method in [[KMc1], Section 6] — this method is applied to more general cases [KMc2]; see also [V2] (Theorem 12.34) for a modified proof. Loeper’s original proof in Section 4 of [Lo] uses the main result of [TW1].

¹⁰The name **DASM** is an abbreviation of “Double mountain above Sliding Mountain” [KMc1] [KMc2]. Villani calls this property *weak regularity* [V2].

(\Leftarrow) This direction is shown for more general case by Loeper [Lo] using Taylor expansion argument. \square

3. THE KEY IDEA OF COUNTEREXAMPLE

In this section, we demonstrate our key idea of the counterexample which we shall construct in Theorem 5.1. We shall find such a situation that **local DASM** is violated for the Riemannian distance squared cost of a nonnegatively curved Riemannian manifold; by Theorem 2.3, **A3w** then shall be violated, too.

Let M be a Riemannian manifold and let c denote the Riemannian distance squared cost function $c(\cdot, \cdot) = \text{dist}^2(\cdot, \cdot)/2$. In the following discussion, we assume that c is differentiable whenever necessary. Let $t \in [0, 1] \rightarrow \bar{x}(t)$, be a c -segment with respect to x . Thus there exist $p, \eta \in T_x M$ such that $\bar{x}(t) = \exp_x(p + t\eta)$ and $c(x, \bar{x}(t)) = |p + t\eta|^2/2$. Choose a tangent vector $\xi \in T_x M$ with $\xi \perp \eta$. This orthogonality shall be crucial. Let $y = \exp_x(s_0\xi)$ for a sufficiently small $s_0 > 0$. Now suppose that there exist a point $\bar{x}(t_0)$ for $1/2 < t_0 < 2/3$ and a sufficiently small open neighborhood B of $\bar{x}(t_0)$ such that the Gaussian curvature K satisfies $K \equiv 0$ on $M \setminus B$ and $K > 0$ on B . (M is nonnegatively curved.) Further assume that the tangent vectors $p + t_0\eta$ and ξ are not collinear. We see that

$$(3.1) \quad \begin{aligned} c(y, \bar{x}(0)) &= |s_0\xi - p|^2/2, \\ c(y, \bar{x}(1)) &= |s_0\xi - p - \eta|^2/2. \end{aligned}$$

Since $K > 0$ near $\bar{x}(t_0)$ and $K \geq 0$ everywhere, by Toponogov's comparison theorem (Theorem 2.1),

$$(3.2) \quad c(y, \bar{x}(t_0)) < |s_0\xi - p - t_0\eta|^2/2.$$

By the orthogonality $\xi \perp \eta$, the function $\tilde{f}(t) = -|s_0\xi - p - t\eta|^2/2 + |p + t\eta|^2/2$ is constant! Thus by (3.1),

$$\begin{aligned} f_0(y) &= -c(y, \bar{x}(0)) + c(x, \bar{x}(0)) \\ &= -|s_0\xi - p|^2/2 + |p|^2/2 \\ &= -|s_0\xi - p - \eta|^2/2 + |p + \eta|^2/2 \\ &= f_1(y), \end{aligned}$$

where $f_t(y) = -c(y, \bar{x}(t)) + c(x, \bar{x}(t))$. By (3.2),

$$\begin{aligned} f_{t_0}(y) &> -|s_0\xi - p - t_0\eta|^2/2 + |p + t_0\eta|^2/2 \\ &= \tilde{f}(t_0) = \tilde{f}(0) = f_0(y) = f_1(y) \\ &= \max[f_0(y), f_1(y)]. \end{aligned}$$

This violates **local DASM**.

4. SOME RESULTS IN RIEMANNIAN GEOMETRY

In the following we prove some technical results (Theorem 4.1 and Corollary 4.2) in Riemannian geometry; these results seem to be new and they are used in our

construction of a nonnegatively curved manifold that does not satisfy **A3w** (see Theorem 5.1).

First recall some definitions (c.f. [CE]). Let M be a Riemannian manifold and x be a point in M . Let σ be a geodesic from x , i.e. $\sigma = \exp_x(t\xi)$, $t \geq 0$, $\xi \in T_x$. A point $y \in M$ is called a *conjugate point* of x along σ if $y = \exp_x(t_0\xi)$ and \exp_x is singular at $t_0\xi$. A point $y \in M$ is called a *cut point* of x if either there are two distinct minimal geodesics from x to y or there is a unique minimal geodesic γ from x to y and y is a conjugate point of x along γ . The *injectivity radius* $\text{inj}_M(x)$ and *conjugate radius* $\text{conj}_M(x)$ at x are defined as follows:

$$\begin{aligned}\text{inj}_M(x) &= \inf\{\text{dist}(x, y) \mid y \text{ is a cut point of } x\}, \\ \text{conj}_M(x) &= \inf\{\text{dist}(x, y) \mid y \text{ is a conjugate point of } x\},\end{aligned}$$

where $\text{dist}(x, y)$ denotes the Riemannian distance between x and y . Note that $\text{inj}_M(x) \leq \text{conj}_M(x)$ and if a geodesic σ from x to z has length less than $\text{inj}_M(x)$ then σ is minimal.

The following result and its corollary are used later in Section 5, but they have their own independent interests.

Theorem 4.1. (injectivity radius = conjugate radius) *Let M be a 2-dimensional simply connected complete Riemannian manifold and let K denote its Gaussian curvature. Suppose $\int_M K_+ d\text{vol} < \pi$ where K_+ denotes the positive part of K , i.e. $K_+ = \max[K, 0]$. Then for every $x \in M$, $\text{inj}_M(x) = \text{conj}_M(x)$.*

Proof. Suppose $\text{inj}_M(x) < \text{conj}_M(x)$. It is easy to see that

- there is a point $y \in M$ such that $\text{dist}(x, y) = \text{inj}_M(x)$;
- there are two distinct minimal geodesics, say $t \in [0, 1] \rightarrow \gamma_i(t) = \exp_x(t\xi_i)$, $i = 0, 1$, from x to y , e.g. $\gamma_0(1) = \gamma_1(1) = y$, $\dot{\gamma}_0(0) \neq \dot{\gamma}_1(0)$;
- \exp_x is non-singular at $\xi_0, \xi_1 \in T_x M$.

First, the tangent vectors $-\dot{\gamma}_0(1)$, $-\dot{\gamma}_1(1)$ at y have exactly the opposite direction, i.e. they form angle π . If this is not the case, then there exists a tangent vector η at y which forms the same angle ($< \pi/2$) with $-\dot{\gamma}_0(1)$ and $-\dot{\gamma}_1(1)$. Because of non-singularity of \exp_x at ξ_0, ξ_1 , there exist two distinct continuous families of geodesics $\{\gamma_i^s\}$, $i = 0, 1$, for $0 < s \ll 1$, such that each γ_i^s connects x to the points $\exp_y(s\eta)$. (In these families $\{\gamma_i^s\}$ the initial velocity vectors of geodesics are close to that of γ_i , $i = 0, 1$, respectively.) By the first variation formula of arc-length (c.f. [CE]) and the angle condition ($< \pi/2$), it is easy to see that for small $0 < s \ll 1$, these geodesics have lengths smaller than the length of γ_i , $i = 0, 1$. This contradicts that the length of γ_i , $i = 0, 1$, is the same as $\text{inj}_M(x)$.

By simple connectedness of M , the geodesics γ_i , $i = 0, 1$, joined together bound a domain D which is a topological disk. Suppose D is oriented in such a way that $\partial D = \gamma_0 - \gamma_1$ (here the parametrization $\gamma_i(t) = \exp_x(t\xi_i)$, $i = 0, 1$, give the orientations of γ_0 and γ_1 .) Let ϑ be the counter-clockwise angle from $-\dot{\gamma}_1(0)$ to $\dot{\gamma}_0(0)$ at x . By Gauss-Bonnet theorem and our assumption $\int_D K < \pi$,

$$2\pi = 2\pi\chi(D) = \vartheta + \int_D K < \pi + \pi$$

which is a contradiction! This completes the proof. \square

Corollary 4.2. (large injectivity radius) *In addition to the assumptions in Theorem 4.1, further assume that $K \leq \delta$ everywhere for a fixed $\delta > 0$. Then, $\text{inj}_M(x) \geq \frac{\pi}{\sqrt{\delta}}$ for every $x \in M$.*

Proof. This follows from Theorem 4.1 and Rauch's comparison theorem (c.f. [CE]). \square

5. NONNEGATIVELY OR POSITIVELY CURVED RIEMANNIAN MANIFOLDS NOT SATISFYING **A3w** CONDITION

In the following, it shall be shown that a shallow, smooth convex cone (which is nonnegatively curved) fits well into the situation in Section 3 and it does not satisfy **local DASM**, thus not **A3w** by Theorem 2.3.

Theorem 5.1. (nonnegative curvature $\not\Rightarrow$ A3w) *A nonnegatively curved complete (open or closed) manifold does not necessarily satisfy A3w.*

Proof. This theorem shall be proven by constructing two complete nonnegatively curved surfaces such that one is open, the other is closed, and both of them do not satisfy **local DASM**. These examples then do not satisfy **A3w** condition (see Theorem 2.3).

Fix cartesian coordinates (a, b) of \mathbf{R}^2 with the origin $O = (0, 0)$. Let $\theta(a, b)$ denote the polar angle of (a, b) with respect to the origin which is counter-clockwise from the positive a -axis. For example, $\theta(1, 0) = 0$ and $\theta(0, 1) = \pi/2$.

Let ϑ be a sufficiently small positive number, i.e. $0 < \vartheta \ll 1$, and define an infinite conical sector \mathcal{C}_ϑ by

$$\mathcal{C}_\vartheta = \mathbf{R}^2 \setminus \{(a, b) \in \mathbf{R}^2 \mid \frac{3\pi}{2} - \vartheta < \theta(a, b) \leq \frac{3\pi}{2} + \vartheta\}.$$

Let B denote $B(O, 1) \cap \mathcal{C}_\vartheta$, where $B(O, 1)$ is the *open* unit disk centered at O . By identifying the two sides of $\partial\mathcal{C}_\vartheta$, we view this domain \mathcal{C}_ϑ as an infinite cone in \mathbf{R}^3 with conical angle $2\pi - 2\vartheta$. It is easy to see that by only perturbing the metric inside B , this cone can be changed to a smooth surface $\Sigma_\vartheta \subset \mathbf{R}^3$, in such a way that

- (1) the Riemannian metric of Σ_ϑ is radially symmetric with respect to the center (the point corresponding to O);
- (2) the Gaussian curvature K of Σ_ϑ as a function on \mathcal{C}_ϑ satisfies $K \equiv 0$ on $\mathcal{C}_\vartheta \setminus B$;
- (3) $0 < K < \frac{1}{10000}$ on B . Here, (3) is possible since $0 < \vartheta \ll 1$.

For later use, it is important to note that by Theorem 4.1 and Corollary 4.2, (2) & (3) imply

- (4) $\text{inj}_{\Sigma_\vartheta}(z) > 314$ for every $z \in \Sigma_\vartheta$.

In the following, the cartesian coordinates of \mathcal{C}_ϑ shall be used to describe points in Σ_ϑ . Let c denote the Riemannian distance squared cost of the surface Σ_ϑ . Let $x = (10, 10)$, $y = (10, 11)$ and let $t \in [0, 1] \rightarrow \bar{x}(t)$ be the c -segment with respect to

x from $\bar{x}(0) = (-10, 1/2)$ to $\bar{x}(1) = (10, 1/2)$, which is just an exponential image (with respect to the metric of Σ_ϑ) of a line segment in the tangent space at x , i.e. $\bar{x}(t) = \exp_x(p + t\eta)$ for $p, \eta \in T_x\Sigma_\vartheta$; moreover, $c(x, \bar{x}(t)) = |p + t\eta|^2/2$, $0 \leq t \leq 1$. Note that our conditions (1), (2), (3), & (4) ensure that c is differentiable for any pair of points inside $B(O, 100) \cap \mathcal{C}_\vartheta$, so for all relevant points in our consideration. These conditions also make it clear that

- each point $\bar{x}(t)$ is connected to x and y by unique minimal geodesics;
- the unique minimal geodesics from x and y to $\bar{x}(0)$, $\bar{x}(1)$ are the straight line segments outside B ;
- the curve $t \in [0, 1] \rightarrow \bar{x}(t)$ coincides with the straight line segment from $\bar{x}(0)$ to $\bar{x}(1)$ outside the ball B .

Therefore, there exists $0 < t_0 < 1$ with $\bar{x}(t_0)$ in B . Thus, by following the same lines of Section 3, Σ_ϑ does not satisfy **local DAMS**. This Σ_ϑ furnishes an example of open nonnegatively curved manifold not satisfying **A3w**.

To get a closed surface example, first cut off a large geodesic ball B_1 of O , e.g. with radius 10000, from the surface Σ_ϑ , then glue a flat disk to B_1 along ∂B_1 and round-off the curve where the disk and B_1 are glued, in such a way that the resulting surface is smooth, radially symmetric from O , and convex (thus, the Gaussian curvature is nonnegative). This completes the proof. \square

Corollary 5.2. (positive curvature $\not\Rightarrow$ A3w) *A positively curved complete (open or closed) manifold does not necessarily satisfy A3w.*

Proof. It is possible to perturb (radially symmetrically) the above nonnegatively curved examples so that the resulting manifolds have positive curvature everywhere. If **local DAMS** is violated at some points in the original manifolds, then it should be violated in the perturbed manifolds as well for sufficiently small perturbations. By Theorem 2.3, the corollary follows. \square

Remark 5.1. (higher dimensional examples) The examples in Theorem 5.1 are radially symmetric and we can easily construct higher dimensional radially symmetric examples in which our 2-dimensional examples are isometrically and totally geodesically embedded. Then the higher dimensional examples do not satisfy **A3w**, neither their positively curved radially symmetric perturbations. One may also consider taking a Riemannian product of a positively curved but non-**A3w** manifold with other positively curved manifold, then certainly the resulting manifold is nonnegatively curved but violates **A3w**; however, it is not clear whether we can perturb the product to a positively curved manifold — it is a famous conjecture of H. Hopf that $S^2 \times S^2$ does not carry a positive curvature metric. See [KMc2] [KMc3] for more considerations on products.

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