Lower bounds of eigenvalues for a class of bi-subelliptic operators

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Abstract

Let $\Omega$ be a bounded open domain in $\mathbb{R}^n$ with smooth boundary and $X = (X_1, X_2, \ldots, X_m)$ be a system of real smooth vector fields defined on $\Omega$ with the boundary $\partial \Omega$ which is non-characteristic for $X$. If $X$ satisfies the Hörmander’s condition, then the vector fields is finitely degenerate and the sum of square operator $\triangle = \sum_{i=1}^m X_i^2$ is a subelliptic operator. Let $\lambda_k$ be the $k$-th eigenvalue for the bi-subelliptic operator $\Delta^2$ on $\Omega$. In this paper, we introduce the generalized Métivier’s condition and study the lower bounds of eigenvalues for the operator $\Delta^2$ on some finitely degenerate systems of vector fields $X$ which satisfy the Hörmander’s condition or the generalized Métivier’s condition. By using the subelliptic estimates, we shall give a simple lower bound estimates of $\lambda_k$ which is polynomial increasing in $k$ with the order relating to the Hörmander index or the generalized Métivier index.

Keywords: eigenvalues, finitely degenerate elliptic operators, Hörmander’s condition, Métivier’s condition, subelliptic estimate, bi-subelliptic operator

1. Introduction and Main Results

For $n \geq 2$, the systems of real smooth vector fields $X = (X_1, X_2, \ldots, X_m)$ defined on an open domain $W$ in $\mathbb{R}^n$. Let $J = (j_1, \ldots, j_l)$ with $1 \leq j_i \leq m$, $X^J = X_{j_1}X_{j_2}X_{j_3} \cdots X_{j_{l-1}}X_{j_l}$, $|J| = l$; and $X^J = id$ if $|J| = 0$. Then We introduce following function space (cf. [21], [24], [27]):

$$H^2_X(W) = \{u \in L^2(W) \mid X^J u \in L^2(W), |J| \leq 2\},$$

which is a Hilbert space with norm $\|u\|^2_{H^2_X(W)} = \sum_{|J| \leq 2} \|X^J u\|^2_{L^2(W)}$.

We say that $X = (X_1, X_2, \ldots, X_m)$ satisfies the Hörmander’s condition on $W$ if there exists a positive integer $Q$, such that for any $|J| = k \leq Q$, $X$ together with all $k$-th repeated commutators $X_J = [X_{j_1}, [X_{j_2}, [X_{j_3}, \ldots, [X_{j_{k-1}}, X_{j_k}] \cdots]]$ span the tangent space at each point of $W$. Here $Q$ is called the Hörmander index of $X$ on $W$, which is defined as the smallest positive integer for the Hörmander’s condition above being satisfied.

Let $\Omega \subset W$ be a bounded open subset with smooth boundary $\partial \Omega$ which is non characteristic for $X$. If $X$ satisfies the Hörmander’s condition on $\Omega$ with $1 \leq Q < +\infty$, then we say that $X$ is a finitely

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degenerate system of vector fields on $\Omega$ and the finitely degenerate elliptic operator $\triangle_X = \sum_{i=1}^{m} X_i^2$ is a subelliptic operator. Let $H^2_{X,0}(\Omega)$ be a subspace defined as a closure of $C_0^\infty(\Omega)$ in $H^2_X(W)$. Then $H^2_{X,0}(\Omega)$ is also a Hilbert space.

In this paper, we consider the following eigenvalue problems of bi-subelliptic operators in $H^2_{X,0}(\Omega)$,
\begin{align}
\begin{cases}
\triangle_X^2 u = \lambda u, & \text{in } \Omega, \\
u = 0, X u = 0, & \text{on } \partial \Omega,
\end{cases}
\end{align}
where $X u$ denotes the gradient $(X_1 u, \cdots, X_m u)$ according to the finitely degenerate system of vector fields $X$.

Before we state our results, we would like to remark on the estimates of eigenvalues for the classical biharmonic operator $\triangle^2$, which is generally called the eigenvalue problem for the clamped plate.

If $X = (\partial_{x_1}, \cdots, \partial_{x_n})$, then $\triangle^2_X$ is the standard biharmonic operator $\triangle^2$ with Hörmander index $Q = 1$. In 1985, Levine and Protter [16] proved that the eigenvalues $\{\lambda_k\}_{k \geq 1}$ of the clamped plate problem
\begin{align}
\begin{cases}
\triangle^2 u = \lambda u, & \text{in } \Omega, \\
u = 0, \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial \Omega,
\end{cases}
\end{align}

satisfy
\begin{align}
\sum_{i=1}^{k} \lambda_i \geq \frac{16\pi^4}{n+4} \left( \frac{\omega_{n-1}|\Omega|_n}{n} \right)^{-\frac{4}{n}} k^{1+\frac{4}{n}},
\end{align}

where $\frac{\partial u}{\partial \nu}$ denotes the derivative of $u$ with respect to the outer unit normal vector $\nu$, and $\{\lambda_k\}_{k \geq 1}$ are eigenvalues for $\triangle^2$, $|\Omega|_n$ is the $n$-dimensional Lebesgue measure of $\Omega$ and $\omega_{n-1}$ denotes the area of the unit ball in $\mathbb{R}^n$.

If $X$ is a finitely degenerate system of vector fields on $\Omega$ with its Hörmander index $1 < Q < +\infty$, then $\triangle_X = \sum_{i=1}^{m} X_i^2$ is a finitely degenerate elliptic operator. For this case, many results are obtained in [2] and [3].

In the first part of this paper, we shall study the general finitely degenerate systems of vector fields $X = (X_1, X_2, \cdots, X_m)$ which satisfy the Hörmander’s condition and get the precise lower bound estimates of the eigenvalues for the problem (1.1).

In the finitely degenerate case, we know that there is a sequence of discrete eigenvalues $\{\lambda_k\}_{k \geq 1}$ for the problem (1.1) satisfying $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \cdots$ and $\lambda_k \to +\infty$ as $k \to +\infty$. By using the similar approach in the work of Li and Yau [17], we can prove the following simple lower bounds of eigenvalues problem for the bi-subelliptic operator $\triangle^2_X$.

**Theorem 1.1.** If $X = (X_1, X_2, \cdots, X_m)$ satisfies the Hörmander’s condition with its Hörmander index $1 \leq Q < +\infty$, $\lambda_i$ is the $i$-th eigenvalue of the problem (1.1), then for all $k \geq 1$
\begin{align}
\sum_{i=1}^{k} \lambda_i \geq C k^{1+\frac{4}{Q}} - \frac{\tilde{C}(Q)}{C(Q)} k
\end{align}
with
\[ C = \frac{n^{1+\frac{4}{nQ}}(2\pi)^{\frac{4}{nQ}}}{C(Q)(nQ+4)|\Omega|_{n}\omega_{n-1}^{\frac{4}{nQ}}}, \]
where \( C(Q), \tilde{C}(Q) \) are the constants in the subelliptic estimates (2.1) below, \( \omega_{n-1} \) is the area of the unit sphere in \( \mathbb{R}^n \), \( |\Omega|_{n} \) is the volume of \( \Omega \).

**Remark 1.1.**

1. Since \( k\lambda_k \geq \sum_{i=1}^{k} \lambda_i \), then Theorem 1.1 shows that the eigenvalues \( \lambda_k \) satisfy
\[ \lambda_k \geq Ck^{\frac{4}{nQ}} - \frac{C(Q)}{C(Q)} \tilde{C}(Q), \] for all \( k \geq 1 \).

2. If \( \Delta^2_X = \Delta^2 \) is the standard biharmonic operator, then \( Q=1, C(Q) = 1, \tilde{C}(Q) = 0, C = \frac{16\pi^4}{n+4} \left( \frac{\omega_{n-1} |\Omega|_{n}^{4/n}}{n} \right)^{-\frac{4}{n}} \). Thus the result of Theorem 1.1 is the same as the estimate of Levine and Protter [16].

Furthermore, if \( X \) satisfies the Hörmander’s condition on \( \Omega \) with the Hörmander index \( Q \), then for each \( 1 \leq j \leq Q \) and \( x \in \Omega \), we denote \( V_j(x) \) as the subspace of the tangent space \( T_x(\Omega) \) which is spanned by the vector fields \( X_j \) with \( |J| \leq j \) (by convention the \( X_j \) themselves will be regarded as commutators of length one). If the dimension of \( V_j(x) \) is constant \( v_j \) in a neighborhood of each \( x \in \Omega \), then we say the system of the vector fields \( X \) satisfies the so called Métivier’s condition on \( \Omega \) and the Métivier index is defined as
\[ v = \sum_{j=1}^{Q} j(v_j - v_{j-1}), \] here \( v_0 = 0 \), (1.6)

where \( v \) is also called the Hausdorff dimension of \( \Omega \) related to the subelliptic metric induced by the vector fields \( X \).

In 1976, Métivier proved an asymptotic result for the problem (1.1) under the Métivier’s condition
\[ \lambda_k \approx k^{\frac{2}{v}}, \] as \( k \to +\infty \), (1.7)

where \( v \) is the Métivier index above. Indeed, for the general case in Theorem 1.1 we can only deduce that the increasing order of \( k \) in the lower bound is \( \frac{4}{nQ} \), which is not greater than \( \frac{4}{\tilde{v}} \), and there is still no certain lower bounds result which is related to the Métivier index. But we can obtain a higher increasing order for the lower bounds under the generalized Métivier’s condition as follows, which is discussed in [2].

Next we introduce the following generalized Métivier index. By using the same notation we denote here \( v_j(x) = \dim V_j(x), v(x) = \sum_{j=1}^{Q} j(v_j(x) - v_{j-1}(x)), \) with \( v_0(x) = 0 \). Then we define
\[ \tilde{v} = \max_{x \in \Omega} v(x) \] (1.8)
as the generalized Métivier index. Thus a degenerate vector fields \( X \) always has the generalized Métivier index \( \tilde{v} \) on \( \Omega \) even if the Métivier’s condition will be not satisfied for \( X \). Observe \( \tilde{v} = v \) if the Métivier’s condition is satisfied.
In the second part, we shall study the bi-subelliptic operators $\triangle_X^2$ on two kinds of Grushin-type vector fields which satisfy the generalized Métivier’s condition and get the precise lower bound estimates for eigenvalues.

**Theorem 1.2.** Let $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, x_1^p \partial_{x_n})$ and $\Omega \cap \{x_1 = 0\} \neq \emptyset$. Then $X$ is a Grushin-type system of vector fields on $\Omega$ with the $n$-th degenerate direction and the Hörmander index $Q=p+1$. Also the generalized Métivier index $\tilde{v} = n + Q - 1$. Suppose $\lambda_i$ be the $i$-th eigenvalue of the problem \((1.1)\), then

$$\sum_{i=1}^{k} \lambda_i \geq \tilde{C}(Q) k^{1+\frac{4}{n}} - \frac{C_2(Q)}{C_1(Q)} k,$$

(1.9)

where

$$\tilde{C}(Q) = \frac{A_Q}{C_1(Q)n^2(n+Q+3)} \left(\frac{(2\pi)^n}{Q\omega_{n-1}|\Omega|_n}\right)^{\frac{4}{n+Q-1}} (n+Q-1)^{\frac{n+Q+3}{n+Q-1}},$$

and

$$A_Q = \begin{cases} \min\{1, n^{-\frac{2}{n}}\}, & Q \geq 2, \\ n, & Q = 1; \end{cases}$$

$c_1, c_2(Q)$ are the constants in Proposition 2.3 below, $\omega_{n-1}$ is the area of the unit sphere in $\mathbb{R}^n$, $|\Omega|_n$ is the volume of $\Omega$.

**Remark 1.2.** (1) Since $k\lambda_k \geq \sum_{i=1}^{k} \lambda_i$, then Theorem 1.2 shows that the eigenvalues $\lambda_k$ satisfy

$$\lambda_k \geq \tilde{C}(Q) k^{\frac{4}{n}} - \frac{C_2(Q)}{C_1(Q)} k,$$

(1.10)

(2) If $\triangle_X^2 = \triangle^2$ is the standard biharmonic operator, then $Q=1$, $C_1(Q) = 1$, $C_2(Q) = 0$, $\tilde{C}(Q) = \frac{16\pi n}{n+4} \left(\frac{\omega_{n-1}|\Omega|_n}{n}\right)^{-\frac{4}{n}}$. Thus the result of Theorem 1.2 is the same as the estimate of Levine and Protter [10].

**Theorem 1.3.** Let $X = (\partial_{x_1}, \cdots, x_i^p \partial_{x_{n-1}}, x_j^q \partial_{x_n})$, $n \geq 3$, $p, q \in \mathbb{Z}_+$, $i, j \in \{1, 2, \cdots, n-2\}$, $X$ is a Grushin-type system of vector fields on $\Omega$ with the $(n-1)$-th and $n$-th degenerate directions. If $\Omega \cap \{x_i = 0\} \neq \emptyset$ and $\Omega \cap \{x_j = 0\} \neq \emptyset$. Then $X$ satisfies the Hörmander’s condition on $\Omega$ with $Q = \max\{p, q\} + 1$ and its generalized Métivier index $\tilde{v} = n + p + q$. Suppose $\lambda_i$ be the $i$-th eigenvalue of the problem (1.1), then

$$\sum_{i=1}^{k} \lambda_i \geq \tilde{C}(p, q) k^{1+\frac{4}{n}} - \frac{C_2(p, q)}{C_1(p, q)} k,$$

(1.11)

with

$$\tilde{C}(p, q) = \frac{2^n}{5C_1(p, q)n^{\frac{n+p+q+n}{2}}} \left(\frac{n + p + q}{(p+1)(q+1)\omega_{n-1}}\right)^{1+\frac{4}{n+p+q}} \left(\frac{(2\pi)^n}{|\Omega|_n}\right)^{\frac{4}{n+p+q}},$$

where $C_1(p, q)$ and $C_2(p, q)$ are the corresponding subelliptic estimate constants in proposition 2.4.
Remark 1.3. Since \( k\lambda_k \geq \sum_{i=1}^{k} \lambda_i \), then Theorem 1.3 shows that the eigenvalues \( \lambda_k \) satisfy
\[
\lambda_k \geq \tilde{C}(p,q)k^{\frac{q}{p}} - \frac{C_2(p,q)}{C_1(p,q)}, \text{ for all } k \geq 1.
\] (1.12)

Our paper is organized as follows. In Section 2, we introduce some preliminaries about subelliptic estimate and maximally hypoellipticity. In section 3, we prove Theorem 1.1. In section 4, we prove Theorem 1.2. Finally, we prove Theorem 1.3 in section 5.

2. Preliminaries

Firstly, we introduce the subelliptic estimates as follows.

Proposition 2.1. If the system of vector fields \( X = (X_1, \cdots, X_m) \) satisfies the Hörmander’s condition on \( \Omega \) with its Hörmander index \( Q \geq 1 \), if and only if the following subelliptic estimate
\[
\left\| \nabla |\hat{\sigma}|^2 u \right\|_{L^2(\Omega)}^2 \leq C(Q) \left\| \Delta_X u \right\|_{L^2(\Omega)}^2 + \tilde{C}(Q) \left\| u \right\|_{L^2(\Omega)}^2
\] (2.1)
holds for all \( u \in C_0^\infty(\Omega) \). Where \( \nabla = (\partial_{x_1}, \ldots, \partial_{x_m}) \), \( |\hat{\sigma}|^2 \) is a pseudo-differential operator with the symbol \( |\xi|^2 \), the constants \( C(Q) > 0, \tilde{C}(Q) \geq 0 \) depending on \( Q \).

Proof. Refer to [11] and [24], the subelliptic operator \( \Delta_X = \sum_{i=1}^{m} X_i^2 \) satisfies
\[
\|u\|_{(2\epsilon)} \leq C_1 \| \Delta_X u \|_{L^2(\Omega)} + C_2 \| u \|_{L^2(\Omega)}
\]
with \( \epsilon = \frac{1}{Q} \), where \( \|u\|_{(2\epsilon)} \) is the Sobolev norm of order \( 2\epsilon \). On the other hand, we have
\[
\|u\|_{(\frac{Q}{2})} = \left( \int_{\mathbb{R}^n} \left( 1 + |\xi|^2 \right)^{\frac{Q}{2}} |\hat{\sigma}(\xi)|^2 d\xi \right)^{\frac{1}{2}}
\]
\[
\geq \left( \int_{\mathbb{R}^n} |\xi|^2 |\hat{\sigma}(\xi)|^2 d\xi \right)^{\frac{1}{2}}
\]
\[
= \left\| \nabla |\hat{\sigma}|^2 u \right\|_{L^2(\mathbb{R}^n)} = \left\| \nabla |\hat{\sigma}|^2 u \right\|_{L^2(\Omega)}.
\]
Using the Cauchy-Schwarz inequality we get the estimate
\[
\left\| \nabla |\hat{\sigma}|^2 u \right\|_{L^2(\Omega)}^2 \leq C(Q) \left\| \Delta_X u \right\|_{L^2(\Omega)}^2 + \tilde{C}(Q) \left\| u \right\|_{L^2(\Omega)}^2
\]

Proposition 2.2. (cf. [22], [27] and [23]) If the system of vector fields \( X = (X_1, \cdots, X_m) \) satisfies the Hörmander’s condition at any point of \( \Omega \), then the operator \( \Delta_X = \sum_{i=1}^{m} X_i^2 \) is maximally hypoelliptic, i.e., there exists a constant \( C > 0 \) such that
\[
\sum_{|\alpha| \leq 2} \| X^\alpha u \|_{L^2(\Omega)}^2 \leq C(\| \Delta_X u \|_{L^2(\Omega)}^2 + \| u \|_{L^2(\Omega)}^2), \forall u \in C_0^\infty(\Omega),
\]
where \( \alpha = (\alpha_1, \cdots, \alpha_m) \) is a multi-index with \( |\alpha| = \alpha_1 + \cdots + \alpha_m \) and \( X^\alpha = X_1^{\alpha_1} \cdots X_m^{\alpha_m} \).
Proposition 2.3. If \( X = (X_1, X_2, \cdots, X_{n-1}, X_n) = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_{n-1}}, x_1^p \partial_{x_n}) \) and its Hörmander index \( Q = p + 1 \), then we have the following subelliptic estimate

\[
\sum_{i=1}^{n-1} \| \partial_{x_i}^2 u \|^2_{L^2(\Omega)} + \| \partial_{x_n} \frac{\partial}{\partial x_1} u \|^2_{L^2(\Omega)} \leq C_1(Q) \| \Delta X u \|^2_{L^2(\Omega)} + C_2(Q) \| u \|^2_{L^2(\Omega)},
\]

(2.2)

for all \( u \in C_0^\infty(\Omega) \). Where \( |\partial_{x_n} \frac{\partial}{\partial x_1} u | \) is a pseudo-differential operator with the symbol \( |\xi_n|^\frac{2}{Q} \), \( C_1(Q) > 0 \), \( C_2(Q) \geq 0 \) are constants depending on \( Q \).

Proof. From the Plancherel’s formula, we have

\[
\| \partial_{x_n} \frac{\partial}{\partial x_1} u \|^2_{L^2(\Omega)} = \| \xi_n \frac{\partial}{\partial x_1} u \|^2_{L^2(\mathbb{R}^n)} \leq \| \nabla \frac{\partial}{\partial x_1} u \|^2_{L^2(\mathbb{R}^n)} = \| \nabla \frac{\partial}{\partial x_1} u \|^2_{L^2(\Omega)}.
\]

(2.3)

Also, from the maximally hypoelliptic estimate of Proposition 2.2 we can deduce that

\[
\sum_{i=1}^{n-1} \| \partial_{x_i}^2 u \|^2_{L^2(\Omega)} \leq \sum_{|\alpha| \leq 2} \| X^\alpha u \|^2_{L^2(\Omega)} \leq C(\| \Delta X u \|^2_{L^2(\Omega)} + \| u \|^2_{L^2(\Omega)}).
\]

(2.4)

Combining (2.1), (2.3) and (2.4) we can deduce that

\[
\sum_{i=1}^{n-1} \| \partial_{x_i}^2 u \|^2_{L^2(\Omega)} + \| \partial_{x_n} \frac{\partial}{\partial x_1} u \|^2_{L^2(\Omega)} \leq C_1(Q) \| \Delta X u \|^2_{L^2(\Omega)} + C_2(Q) \| u \|^2_{L^2(\Omega)}.
\]

(2.5)

Proposition 2.4. If \( X = (X_1, X_2, \cdots, X_{n-1}, X_n) = (\partial_{x_1}, \partial_{x_2}, \cdots, x_i \partial_{x_{n-1}}, x_j \partial_{x_n}) \), \( n \geq 3 \), \( p, q \in \mathbb{Z}_+ \), \( i, j \in \{1, 2, \cdots, n-2\} \), then we have the following subelliptic estimate

\[
\sum_{i=1}^{n-2} \| \partial_{x_i}^2 u \|^2_{L^2(\Omega)} + \| \partial_{x_{n-1}} \frac{\partial}{\partial x_1} u \|^2_{L^2(\Omega)} + \| \partial_{x_n} \frac{\partial}{\partial x_i} u \|^2_{L^2(\Omega)} \leq C_1(p, q) \| \Delta X u \|^2_{L^2(\Omega)} + C_2(p, q) \| u \|^2_{L^2(\Omega)},
\]

(2.5)

for all \( u \in C_0^\infty(\Omega) \). Where \( |\partial_{x_n} \frac{\partial}{\partial x_1} u | \) is a pseudo-differential operator with the symbol \( |\xi_n|^\frac{2}{Q} \), \( C_1(p, q) = C_1(p+1) + C_1(q+1) \), \( C_2(p, q) = C_2(p+1) + C_2(q+1) \), \( C_1(p+1), C_1(q+1), C_2(p+1), C_2(q+1) \geq 0 \) are the corresponding constants in Proposition 2.3.

Proof. We consider the system of vector fields \( \tilde{X} = (\partial_{x_1}, \cdots, \partial_{x_{n-2}}, x_i \partial_{x_{n-1}}) \) defined on the projection \( \Omega' \) of \( \Omega \) on the direction \( x' = (x_1, \cdots, x_{n-1}) \). Similar to Proposition 2.3, we have

\[
\sum_{i=1}^{n-2} \| \partial_{x_i}^2 u \|^2_{L^2(\Omega')} + \| \partial_{x_{n-1}} \frac{\partial}{\partial x_1} u \|^2_{L^2(\Omega')} \leq C_1(p+1) \| \Delta X u \|^2_{L^2(\Omega')} + C_2(p+1) \| u \|^2_{L^2(\Omega')},
\]
Then we have
\[
\sum_{i=1}^{n-2} \left\| \partial_{x_i}^2 u \right\|^2_{L^2(\Omega)} + \left\| \partial_{x_{n-1}} \frac{2}{p+1} u \right\|^2_{L^2(\Omega)} \leq C_1(p+1) \| \Delta X u \|^2_{L^2(\Omega)} + C_2(p+1) \| u \|^2_{L^2(\Omega)}.
\] (2.6)

Similarly, we can deduce that
\[
\sum_{i=1}^{n-2} \left\| \partial_{x_i}^2 u \right\|^2_{L^2(\Omega)} + \left\| \partial_{x_{n-1}} \frac{2}{q+1} u \right\|^2_{L^2(\Omega)} \leq C_1(q+1) \| \Delta X u \|^2_{L^2(\Omega)} + C_2(q+1) \| u \|^2_{L^2(\Omega)}.
\] (2.7)

Finally, we get the subelliptic estimate from (2.6) and (2.7)
\[
\sum_{i=1}^{n-2} \left\| \partial_{x_i}^2 u \right\|^2_{L^2(\Omega)} + \left\| \partial_{x_{n-1}} \frac{2}{p+1} u \right\|^2_{L^2(\Omega)} + \left\| \partial_{x_{n}} \frac{2}{q+1} u \right\|^2_{L^2(\Omega)} \leq (C_1(p+1) + C_1(q+1)) \| \Delta X u \|^2_{L^2(\Omega)} + (C_2(p+1) + C_2(q+1)) \| u \|^2_{L^2(\Omega)}.
\]

3. Proof of Theorem 1.1

**Lemma 3.1.** For the system of vector fields \( X = (X_1, \cdots, X_m) \), if \( \{\phi_j\}^k_{j=1} \) are the set of orthonormal eigenfunctions corresponding to the eigenvalues \( \{\lambda_j\}^k_{j=1} \). Define
\[
\Phi(x, y) = \sum_{j=1}^{k} \phi_j(x)\phi_j(y).
\]

Then for the partial Fourier transformation of \( \Phi(x, y) \) with respect to the \( x \)-variable,
\[
\hat{\Phi}(z, y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Phi(x, y)e^{-ix\cdot z}dx,
\]

we have
\[
\int_{\Omega} \int_{\mathbb{R}^n} |\hat{\Phi}(z, y)|^2 dzdy = k, \int_{\Omega} \left| \hat{\Phi}(z, y) \right|^2 dy \leq (2\pi)^{-n} |\Omega|^n.
\]

**Proof.** Since
\[
\int_{\mathbb{R}^n} \Phi^2(x, y)dx = \int_{\mathbb{R}^n} \left| \hat{\Phi}(z, y) \right|^2 dz.
\]

By the orthonormality of \( \{\phi_j\}^k_{j=1} \), it follows that
\[
\int_{\Omega} \int_{\mathbb{R}^n} \left| \hat{\Phi}(z, y) \right|^2 dzdy = \int_{\Omega} \int_{\mathbb{R}^n} |\Phi(x, y)|^2 dx dy = \int_{\Omega} \int_{\mathbb{R}^n} |\Phi(x, y)|^2 dx dy = k.
\]
On the other hand,
\[ \int_{\mathbb{R}^n} \left| \hat{\Phi}(z) \right|^2 dy = \int_{\Omega} (2\pi)^{-n} \left| \int_{\mathbb{R}^n} \Phi(x, y) e^{-ix \cdot z} dx \right|^2 dy = \int_{\Omega} (2\pi)^{-n} \left| \int_{\Omega} \Phi(x, y) e^{-ix \cdot z} dx \right|^2 dy. \]

Using the Fourier expansion for the function \( e^{-ix \cdot z} \), i.e.
\[ e^{-ix \cdot z} = \sum_{j=1}^{\infty} a_j(z) \phi_j(x), \quad \text{with} \quad a_j(z) = \int_{\Omega} e^{-ix \cdot z} \phi_j(x) dx. \]

Then we obtain that
\[ \sum_{j=1}^{\infty} |a_j(z)|^2 = \int_{\Omega} |e^{-ix \cdot z}|^2 dx = |\Omega|. \]

Thus
\[ \left| \int_{\Omega} \Phi(x, y) e^{-ix \cdot z} dx \right| \leq \left| \int_{\Omega} \sum_{j=1}^{k} \sum_{l=1}^{\infty} a_l(z) \phi_l(x) \phi_j(x) \phi_j(y) dx \right| = \left| \sum_{j=1}^{k} a_j(z) \phi_j(y) \right|. \]

Using the estimates above, we have
\[ \int_{\Omega} \left| \hat{\Phi}(z, y) \right|^2 dy \leq (2\pi)^{-n} \int_{\Omega} \left| \sum_{j=1}^{k} a_j(z) \phi_j(y) \right|^2 dy = (2\pi)^{-n} \sum_{j=1}^{k} |a_j(z)|^2 \leq (2\pi)^{-n} |\Omega|. \]

\[ \square \]

**Lemma 3.2.** Let \( f \) be a real-valued function defined on \( \mathbb{R}^n \) with \( 0 \leq f \leq M_1 \), and for \( Q \in \mathbb{N}^+ \),
\[ \int_{\mathbb{R}^n} |z|^Q f(z) dz \leq M_2. \]

Then
\[ \int_{\mathbb{R}^n} f(z) dz \leq \left( \frac{nQ + 4}{Q} \right)^{\frac{nQ}{nQ+4}} \left( \frac{M_1 \omega_{n-1}}{n} \right)^{\frac{4}{nQ+4}} M_2^{\frac{nQ}{nQ+4}}, \]
where \( \omega_{n-1} \) is the area of the unit sphere in \( \mathbb{R}^n \).

**Proof.** First, we choose \( R \) such that
\[ \int_{\mathbb{R}^n} |z|^4 g(z) dz = M_2, \]
where
\[ g(z) = \begin{cases} M_1, & |z| \leq R, \\ 0, & |z| > R. \end{cases} \]
Then \(\left(z^{\frac{4}{n}} - R^{\frac{4}{n}}\right)(f(z) - g(z)) \geq 0\). Hence we have

\[
R^{\frac{4}{n}} \int_{\mathbb{R}^n} (f(z) - g(z))dz \leq \int_{\mathbb{R}^n} |z^{\frac{4}{n}}(f(z) - g(z))|dz \leq 0.
\]

That means

\[
\int_{\mathbb{R}^n} f(z)dz \leq \int_{\mathbb{R}^n} g(z)dz. \tag{3.1}
\]

Now we have

\[
M_2 = \int_{\mathbb{R}^n} |z^{\frac{4}{n}}g(z)|dz = M_1 \int_{B_R} |z^{\frac{4}{n}}|dz = M_1 \int_0^R r^{n-1+\frac{4}{n}\omega_{n-1}}dr = \frac{M_1Q\omega_{n-1}R^{n+\frac{4}{n}}}{nQ+4}, \tag{3.2}
\]

where \(B_R = \{z \in \mathbb{R}^n, |z| \leq R\}\), \(\omega_{n-1}\) is the area of the unit sphere in \(\mathbb{R}^n\).

On the other hand, we know that

\[
\int_{\mathbb{R}^n} g(z)dz = \frac{M_1\omega_{n-1}R^n}{n} \tag{3.3}
\]

Finally, (3.1), (3.2), (3.3) imply that

\[
\int_{\mathbb{R}^n} f(z)dz \leq \int_{\mathbb{R}^n} g(z)dz = \left(\frac{nQ+4}{Q}\right)^{\frac{nQ}{nQ+4}} \frac{M_1\omega_{n-1}^\frac{4}{n}}{n} M_2^\frac{nQ}{nQ+4}.
\]

**Proof of Theorem 1.1** let \(\{\lambda_k\}_{k \geq 1}\) be a sequence of the eigenvalues for the problem (1.1), \(\{\phi_k(x)\}_{m \geq 1}\) be the corresponding eigenfunctions, then \(\{\phi_k(x)\}_{k \geq 1}\) constitute an orthonormal basis of the Sobolev space \(H^2_{X,0}(\Omega)\). Let \(\Phi(x,y) = \sum_{j=1}^k \phi_j(x)\phi_j(y)\). By using Plancherel’s formula, we have

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |z^{\frac{4}{n}}\tilde{\Phi}(z,y)|^2dydz = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\nabla|^{\frac{2}{n}}\Phi(x,y)|^2dydx
\]

\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\nabla|^{\frac{2}{n}}\Phi(x,y)|^2dydx. \tag{3.4}
\]

Also, from Proposition 2.1

\[
\left\|\nabla|^{\frac{2}{n}}u\right\|_{L^2(\Omega)}^2 \leq C(Q)\|\Delta_Xu\|_{L^2(\Omega)}^2 + \tilde{C}(Q)\|u\|_{L^2(\Omega)}^2, \tag{3.5}
\]

and combining (3.4) with the subelliptic estimate (3.5), we get the following inequality

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |z^{\frac{4}{n}}\tilde{\Phi}(z,y)|^2dydz \leq C(Q)\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\Delta_X\Phi(x,y)|^2dx dy + \tilde{C}(Q)\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi^2(x,y)dx dy. \tag{3.6}
\]
Next, by using integration-by-parts, we have
\[
\sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \int_{\Omega} \lambda_{i} \phi_{i}(x) \cdot \phi_{i}(x) \, dx = \sum_{i=1}^{k} \int_{\Omega} \Delta_{X} \phi_{i}(x) \cdot \phi_{i}(x) \, dx
\]
\[
= \sum_{i=1}^{k} \int_{\Omega} X(\Delta_{X} \phi_{i}(x)) \cdot X \phi_{i}(x) \, dx = \sum_{i=1}^{k} \int_{\Omega} \Delta_{X} \phi_{i}(x) \cdot \Delta_{X} \phi_{i}(x) \, dx
\]
\[
= \sum_{i=1}^{k} \int_{\Omega} \sum_{i=1}^{k} \lambda_{i} \phi_{i}(x) \cdot \phi_{i}(x) \, dx = \int_{\Omega} \int_{\Omega} \lambda_{i} \phi_{i}(x) \cdot \phi_{i}(x) \, dx
\]
\[
= \int_{\Omega} \int_{\Omega} \lambda_{i} \phi_{i}(x) \cdot \phi_{i}(x) \, dx = \int_{\Omega} \int_{\Omega} \lambda_{i} \phi_{i}(x) \cdot \phi_{i}(x) \, dx
\]
(3.7)

Hence we obtain the following result from Lemma 3.1, (3.6) and (3.7)
\[
\int_{\mathbb{R}^{n}} \int_{\Omega} |z|^{\frac{4}{Q}} |\Phi(z, y)|^{2} \, dydz \leq C(Q) \sum_{i=1}^{k} \lambda_{i} + \tilde{C}(Q) k.
\]

Now we choose
\[
f(z) = \int_{\Omega} \left| \Phi(z, y) \right|^{2} \, dy, \quad M_{1} = (2\pi)^{-n} |\Omega|_{n}, \quad M_{2} = C(Q) \sum_{i=1}^{k} \lambda_{i} + \tilde{C}(Q) k.
\]

From Lemma 3.2 for any \( k \geq 1 \),
\[
k \leq \left( \frac{nQ + 4}{Q} \right)^{\frac{nQ}{n+4}} \left( \frac{(2\pi)^{-n} |\Omega|_{n} \omega_{n-1}}{n} \right)^{\frac{4}{n+4}} \left( C(Q) \sum_{i=1}^{k} \lambda_{i} + \tilde{C}(Q) k \right)^{\frac{nQ}{nQ+4}}.
\]

This means, for any \( k \geq 1 \),
\[
\sum_{i=1}^{k} \lambda_{i} \geq C k^{1 + \frac{4}{nQ}} - \frac{\tilde{C}(Q) k}{C(Q)}
\]
with
\[
C = \frac{n^{1 + \frac{4}{nQ}} Q (2\pi)^{\frac{4}{Q}}}{C(Q) (nQ + 4) (\Omega|_{n} \omega_{n-1})^{\frac{4}{nQ}}}
\]

The proof of Theorem 1.1 is complete.

\[\square\]

4. Proof of Theorem 1.2

Lemma 4.1. Let \( f \) be a real-valued function defined on \( \mathbb{R}^{n} \) with \( 0 \leq f \leq M_{1} \), and for \( Q \in \mathbb{N}^{+} \),
\[
\int_{\mathbb{R}^{n}} \left( \sum_{i=1}^{n-1} z_{i}^{2} + |z_{n}|^{2} \right)^{2} f(z) \, dz \leq M_{2}.
\]
Then
\[ \int_{\mathbb{R}^n} f(z) dz \leq \int_{\mathbb{R}^n} g(z) dz \leq \left( Q M_1 \omega_{n-1} \right)^{\frac{n+Q-1}{n+Q+3}} \left( \frac{n(n+Q+3)}{A_Q} \right)^{\frac{n+Q-1}{n+Q+3}} M_2^{\frac{n+Q-1}{n+Q+3}} \]

where \( \omega_{n-1} \) is the area of the unit sphere in \( \mathbb{R}^n \), and
\[ A_Q = \begin{cases} \min\{1, \frac{n}{n-Q} \}, & Q \geq 2, \\ n, & Q = 1. \end{cases} \]

**Proof.** First, we choose \( R \) such that
\[ \int_{\mathbb{R}^n} \left( \sum_{i=1}^{n-1} z_i^2 + |z_n|^\frac{2}{Q} \right)^2 g(z) dz = M_2, \]
where
\[ g(z) = \begin{cases} M_1, & \sum_{i=1}^{n-1} z_i^2 + |z_n|^\frac{2}{Q} \leq R^2, \\ 0, & \sum_{i=1}^{n-1} z_i^2 + |z_n|^\frac{2}{Q} > R^2. \end{cases} \]

Then \( \left( \sum_{i=1}^{n-1} z_i^2 + |z_n|^\frac{2}{Q} - R^2 \right) (f(z) - g(z)) \geq 0. \) Hence we have
\[ R^2 \int_{\mathbb{R}^n} (f(z) - g(z)) dz \leq \int_{\mathbb{R}^n} \left( \sum_{i=1}^{n-1} z_i^2 + |z_n|^\frac{2}{Q} \right) (f(z) - g(z)) dz \leq 0, \]
then
\[ \int_{\mathbb{R}^n} f(z) dz \leq \int_{\mathbb{R}^n} g(z) dz. \] (4.1)

Now we have
\[ M_2 = \int_{\mathbb{R}^n} \left( \sum_{i=1}^{n-1} z_i^2 + |z_n|^\frac{2}{Q} \right)^2 g(z) dz = M_1 \int_{\tilde{B}_R} \left( \sum_{i=1}^{n-1} z_i^2 + |z_n|^\frac{2}{Q} \right)^2 dz, \]
where
\[ \tilde{B}_R = \left\{ z \in \mathbb{R}^n, \sum_{i=1}^{n-1} z_i^2 + |z_n|^\frac{2}{Q} \leq R^2 \right\}, \quad B_R = \left\{ z \in \mathbb{R}^n, \ |z| \leq R \right\}. \]

By using the change of variables,
\[ z_i = z_i' \quad (i = 1, 2, \cdots, n-1), \quad z_n = |z_n'|^Q, \]
then the determinant of Jacobian
\[ \det(\frac{\partial (z_1, \cdots, z_n)}{\partial (z_1', \cdots, z_n')}) = Q |z_n'|^{Q-1}. \]
Hence
\[ M_2 = M_1 \int_{B_R} \left( \sum_{i=1}^{n-1} z_i^2 + |z_n|^2 \right)^2 \, dz = M_1 Q \int_{B_R} |z|^4 |z_n|^{Q-1} \, dz \]
\[ = \frac{M_1 Q}{n} \int_{B_R} |z|^4 \sum_{i=1}^{n} |z_i|^{Q-1} \, dz. \]

On the other hand,
\[
\sum_{i=1}^{n} |z_i|^{Q-1} = |z|^{Q-1} \sum_{i=1}^{n} \left( \frac{|z_i|}{|z|} \right)^{Q-1} \geq A_Q |z|^{Q-1},
\]
where
\[ A_Q = \begin{cases} 
\min\{1, n^{\frac{3-Q}{2}}\}, & Q \geq 2, \\
\frac{n}{Q}, & Q = 1.
\end{cases} \]

Then we have
\[ M_2 \geq \frac{M_1 Q A_Q}{n} \int_{B_R} |z|^{Q+3} \, dz = \frac{M_1 Q A_Q \omega_{n-1}}{n(n + Q + 3)} R^{n+Q+3}. \quad (4.2) \]

From the definition of \( g(z) \), we know that
\[
\int_{\mathbb{R}^n} g(z) \, dz = M_1 \int_{B_R} d\hat{z} = M_1 Q \int_{B_R} |z_n|^{Q-1} \, dz 
\leq M_1 Q \int_{B_R} |z|^{Q-1} \, dz = \frac{M_1 Q \omega_{n-1}}{n + Q - 1} R^{n+Q-1}. \quad (4.3)
\]

Combining (4.1), (4.2) and (4.3), we obtain
\[
\int_{\mathbb{R}^n} f(z) \, dz \leq \int_{\mathbb{R}^n} g(z) \, dz \leq \left( Q M_1 \omega_{n-1} \right)^{n+Q+3} \left( \frac{n(n + Q + 3)}{A_Q} \right)^{\frac{n+Q-1}{n+Q+3}} M_2^{\frac{n+Q-1}{n+Q+3}}.
\]

Lemma 4.1 is proved. \( \square \)

**Proof of Theorem 1.2** Let \( \{\lambda_k\}_{k \geq 1} \) be a sequence of the eigenvalues for the problem (1.1), \( \{\phi_k(x)\}_{m \geq 1} \) be the corresponding eigenfunctions, then \( \{\phi_k(x)\}_{m \geq 1} \) constitute an orthonormal basis of the Sobolev space \( H^2_{X,0}(\Omega) \).

Let \( \Phi(x,y) = \sum_{j=1}^{k} \phi_j(x) \phi_j(y) \), we use Cauchy-Schwarz inequality to get
\[
\int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{i=1}^{n-1} z_i^2 + |z_n|^2 \right)^2 \left| \Phi(z,y) \right|^2 \, dydz
\leq n \int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{i=1}^{n-1} z_i^4 + |z_n|^4 \right) \left| \Phi(z,y) \right|^2 \, dydz. \quad (4.4)
\]
Similar to the result of (3.7), we can deduce that
\[ \sum_{i=1}^{k} \lambda_i = \int_{\Omega} \int_{\Omega} |\Delta_{X} \Phi(x, y)|^2 \, dx \, dy. \]  
(4.5)

Then by using Plancherel’s formula and Proposition 2.3, we have
\[
\int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{4}{n}} \right)^2 |\hat{\Phi}(z, y)|^2 \, dy \, dz
\leq n \int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{i=1}^{n-1} \left| \frac{\partial^2}{\partial x_i} \Phi(x, y) \right|^2 + \left| \frac{\partial}{\partial x_n} \frac{2}{\sqrt{n}} \Phi(x, y) \right|^2 \right) \, dx \, dy
\]
\[
= n \int_{\Omega} \int_{\Omega} \left( \sum_{i=1}^{n-1} \left| \frac{\partial^2}{\partial x_i} \Phi(x, y) \right|^2 + \left| \frac{\partial}{\partial x_n} \frac{2}{\sqrt{n}} \Phi(x, y) \right|^2 \right) \, dy \, dx
\]
\[
= n \int_{\Omega} \int_{\Omega} \left( \sum_{i=1}^{n-1} \left| \frac{\partial^2}{\partial x_i} \Phi(x, y) \right|^2 + \left| \frac{\partial}{\partial x_n} \frac{2}{\sqrt{n}} \Phi(x, y) \right|^2 \right) \, dy \, dx
\]
\[
\leq n \left[ C_1(Q) \int_{\Omega} \int_{\Omega} |\Delta_{X} \Phi(x, y)|^2 \, dx \, dy + C_2(Q) \int_{\Omega} \int_{\Omega} |\Phi(x, y)|^2 \, dx \, dy \right].
\]  
(4.6)

Thus from (4.5) and Lemma 3.1 above, we can deduce that
\[
\int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{4}{n}} \right)^2 |\hat{\Phi}(z, y)|^2 \, dy \, dz \leq n \left( C_1(Q) \sum_{i=1}^{k} \lambda_i + C_2(Q)k \right).
\]

Next, we choose
\[
f(z) = \int_{\Omega} |\hat{\Phi}(z, y)|^2 \, dy, \quad M_1 = (2\pi)^{-n} |\Omega|, \quad M_2 = n \left( C_1(Q) \sum_{i=1}^{k} \lambda_i + C_2(Q)k \right).
\]

Then from the result of Lemma 4.1, we know that for any \( k \geq 1, \)
\[
k \leq \frac{Q_{\omega_{n-1}} (2\pi)^{-n} |\Omega|}{n + Q - 1} \left( \frac{n(n + Q + 3)}{(2\pi)^{-n} |\Omega| Q A Q^{-1} \omega_{n-1}} \right)^{\frac{n+Q-1}{n+Q-3}} \left( n \left( C_1(Q) \sum_{i=1}^{k} \lambda_i + C_2(Q)k \right) \right)^{\frac{n+Q-1}{n+Q-3}}.
\]

This means, for any \( k \geq 1, \)
\[
\sum_{i=1}^{k} \lambda_i \geq \tilde{C}(Q) k^{1+\frac{4}{n}} - \frac{C_2(Q)}{C_1(Q)}k
\]
with
\[
\tilde{C}(Q) = \frac{A_Q}{C_1(Q) n^2 (n + Q + 3)} \left( \frac{(2\pi)^n}{Q \omega_{n-1} |\Omega|} \right)^{\frac{4}{n+Q-1}} (n + Q - 1)^{\frac{n+Q-3}{n+Q-1}}.
\]

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and

\[ A_Q = \begin{cases} 
\min\{1, n, n^{\frac{n-Q}{2}}\}, & Q \geq 2, \\
n, & Q = 1;
\end{cases} \]

The proof of Theorem 1.2 is complete.

5. Proof of Theorem 1.3

Lemma 5.1. Let \( f \) be a real-valued function defined on \( \mathbb{R}^n \) with \( 0 \leq f \leq M_1 \), and for \( p, q \in \mathbb{N}^+ \),

\[
\int_{\mathbb{R}^n} \left( \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^2 + |z_n|^2 \right)^2 f(z) dz \leq M_2.
\]

Then

\[
\int_{\mathbb{R}^n} f(z) dz \leq \frac{(p+1)(q+1)\omega_{n-1}}{n + p + q} M_1^{\frac{4}{n + p + q + 4}} \left( \frac{5n^{\frac{n+p+q+4}{2}}}{2^{n+2}} \right)^{\frac{n+p+q}{n + p + q + 4}} M_2^{\frac{n+p+q}{n + p + q + 4}},
\]

where \( \omega_{n-1} \) is the area of the unit sphere in \( \mathbb{R}^n \).

Proof. First, we choose \( R \) such that

\[
\int_{\mathbb{R}^n} \left( \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^2 + |z_n|^2 \right)^2 g(z) dz = M_2,
\]

where

\[
g(z) = \begin{cases} 
M_1, & \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^2 + |z_n|^2 \leq R^2, \\
0, & \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^2 + |z_n|^2 > R^2.
\end{cases}
\]

Then \( \left( \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^2 + |z_n|^2 - R^2 \right) (f(z) - g(z)) \geq 0 \). Hence we have

\[
R^2 \int_{\mathbb{R}^n} (f(z) - g(z)) dz \leq \int_{\mathbb{R}^n} \left( \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^2 + |z_n|^2 \right) (f(z) - g(z)) dz \leq 0,
\]

that means

\[
\int_{\mathbb{R}^n} f(z) dz \leq \int_{\mathbb{R}^n} g(z) dz. \tag{5.1}
\]

Now we have

\[
M_2 = \int_{\mathbb{R}^n} \left( \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^2 + |z_n|^2 \right)^2 g(z) dz = M_1 \int_{B_R} \left( \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^2 + |z_n|^2 \right)^2 dz,
\]

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where 
\[
\tilde{B}_R = \left\{ z \in \mathbb{R}^n, \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{2+q} + |z_n|^{2+q} \leq R^2 \right\}.
\]

Now we change the variables as follows,
\[
z_i = z'_i \quad (i = 1, 2, \cdots, n-2), \quad z_{n-1} = |z_{n-1}|^{p+1}, \quad z_n = |z_n|^{q+1},
\]
then we have the following determinant of Jacobian,
\[
\det(\frac{\partial(z_1, \cdots, z_n)}{\partial(z'_1, \cdots, z'_n)}) = (p+1)(q+1)|z_{n-1}|^p|z_n|^q.
\]

Hence
\[
M_2 = M_1 \int_{\tilde{B}_R} \left( \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{2+q} + |z_n|^{2+q} \right) dz
\]
\[= M_1 (p+1)(q+1) \int_{B_R} |z|^4|z_{n-1}|^p|z_n|^q dz
\]
\[\geq M_1 (p+1)(q+1) \int_{A_R} |z|^4|z_{n-1}|^p|z_n|^q dz
\]
where
\[B_R = \{ z \in \mathbb{R}^n, |z| \leq R \}, \quad A_R = \left\{ z \in \mathbb{R}^n, |z_i| \leq \frac{R}{\sqrt{n}}, \quad i = 1, \cdots, n \right\}.
\]

On the other hand,
\[
\int_{A_R} |z|^4|z_{n-1}|^p|z_n|^q dz \geq \int_{A_R} |z_1|^4|z_{n-1}|^p|z_n|^q dz = \frac{2^n}{5(p+1)(q+1)n} n^{-\frac{n+p+q+4}{2}} R^{n+p+q+4}.
\]

Then we have
\[
M_2 \geq \frac{2^n M_1}{5} n^{-\frac{n+p+q+4}{2}} R^{n+p+q+4}. \tag{5.2}
\]

From the definition of \(g(z)\), we know that
\[
\int_{\mathbb{R}^n} g(z) dz = M_1 \int_{B_R} dz = M_1 (p+1)(q+1) \int_{B_R} |z_{n-1}|^p|z_n|^q dz
\]
\[\leq \int_{B_R} |z|^{p+q} dz = M_1 (p+1)(q+1) \omega_{n-1} R^{n+p+q}. \tag{5.3}
\]

From (5.1), (5.2) and (5.3), we obtain
\[
\int_{\mathbb{R}^n} f(z) dz \leq \frac{(p+1)(q+1) \omega_{n-1}}{n+p+q} M_1^{\frac{4}{n+p+q+4}} \left( \frac{5n^{\frac{n+p+q+4}{2}}}{2^n} \right)^{\frac{n+p+q}{n+p+q+4}} M_2^{\frac{n+p+q}{n+p+q+4}}.
\]

Lemma 5.1 is proved. \(\square\)
Proof of Theorem 1.3 Let \( \{ \lambda_k \}_{k \geq 1} \) be a sequence of the eigenvalues for the problem (1.1), \( \{ \phi_k(x) \}_{k \geq 1} \) be the corresponding eigenfunctions, then \( \{ \phi_k(x) \}_{k \geq 1} \) constitute an orthonormal basis of the Sobolev space \( H^2_{X,0}(\Omega) \).

Let \( \Phi(x, y) = \sum_{j=1}^{k} \phi_j(x)\phi_j(y) \). Thus, by using the Cauchy-Schwarz inequality, we have

\[
\int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{2/p_{n+1}} + |z_n|^{2/p_{n+1}} \right)^2 |\hat{\Phi}(z, y)|^2 \, dydz \\
\leq n \int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{2/p_{n+1}} + |z_n|^{2/p_{n+1}} \right) |\hat{\Phi}(z, y)|^2 \, dydz.
\]

(5.4)

Similar to the result of (3.7), we obtain that

\[
\sum_{i=1}^{k} \lambda_i = \int_{\Omega} \int_{\Omega} |\triangle_X \Phi(x, y)|^2 \, dx dy.
\]

(5.5)

Then by using Plancherel’s formula and Proposition 2.4, we have

\[
\int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{2/p_{n+1}} + |z_n|^{2/p_{n+1}} \right)^2 |\hat{\Phi}(z, y)|^2 \, dydz \\
\leq n \int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{i=1}^{n-2} |\partial_{x_i}^2 \Phi(x, y)|^2 + \left| |\partial_{x_{n-1}}|^{2/p_{n+1}} \Phi(x, y) \right|^2 + \left| |\partial_{x_n}|^{2/p_{n+1}} \Phi(x, y) \right|^2 \right) \, dydx \\
= n \int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{i=1}^{n-2} |\partial_{x_i}^2 \Phi(x, y)|^2 + \left| |\partial_{x_{n-1}}|^{2/p_{n+1}} \Phi(x, y) \right|^2 + \left| |\partial_{x_n}|^{2/p_{n+1}} \Phi(x, y) \right|^2 \right) \, dydx \\
\leq n \left[ C_1(p, q) \int_{\Omega} \int_{\Omega} |\triangle_X \Phi(x, y)|^2 \, dx dy + C_2(p, q) \int_{\Omega} \int_{\Omega} |\Phi(x, y)|^2 \, dx dy \right].
\]

Thus from (5.5) and Lemma 3.1 above, we can deduce that

\[
\int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{2/p_{n+1}} + |z_n|^{2/p_{n+1}} \right)^2 |\hat{\Phi}(z, y)|^2 \, dydz \\
\leq n \left( C_1(p, q) \sum_{i=1}^{k} \lambda_i + C_2(p, q) k \right).
\]

Finally, we choose

\[
f(z) = \int_{\Omega} |\hat{\Phi}(z, y)|^2 \, dy, \quad M_1 = (2\pi)^{-n}|\Omega|_n, \quad M_2 = n \left( C_1(p, q) \sum_{i=1}^{k} \lambda_i + C_2(p, q) k \right).
\]

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Then from the Lemma 5.1, we have for any \( k \geq 1, \)
\[
k \leq \frac{(p+1)(q+1)\omega_{n-1}}{n+p+q} \left( (2\pi)^{-n} |\Omega|_n \right)^{\frac{4}{n+p+q+4}} \left( \frac{5n}{2} \right)^{\frac{n+p+q}{n+p+q+4}} \]
\[
\times \left( n \left( C_1(p,q) \sum_{i=1}^{k} \lambda_i + C_2(p,q)k \right) \right)^{\frac{4}{n+p+q+4}}.
\]

This means, for any \( k \geq 1, \)
\[
\sum_{i=1}^{k} \lambda_i \geq \tilde{C}(p,q)k^{1+\frac{4}{n+p+q}} - \frac{C_2(p,q)}{C_1(p,q)} k
\]
with
\[
\tilde{C}(p,q) = \frac{2^n}{5C_1(p,q)n^{\frac{n+p+q+4}{n+p+q+4}}} \left( \frac{n+p+q}{(p+1)(q+1)\omega_{n-1}} \right)^{1+\frac{4}{n+p+q}} \left( \frac{2\pi^n}{|\Omega|_n} \right)^{\frac{4}{n+p+q+4}},
\]
and the constant \( C_1(p,q) = C_1(p+1) + C_1(q+1) > 0, C_2(p,q) = C_2(p+1) + C_2(q+1) \geq 0. \)
\( C_1(p+1), C_1(q+1) > 0 \) and \( C_2(p+1), C_2(q+1) \geq 0 \) are the corresponding subelliptic estimate constants in Proposition 2.3. The proof of Theorem 1.3 is complete.

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References


