Research Statement

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1 Research Summary

My current research interest is the study of partial differential equations (PDEs). In particular, I am interested in discovering new type of solutions for several classical non-linear elliptic partial differential equations arising from physics and biology, and studying various properties of these solutions like non-degeneracy, uniqueness....

2 My Research Results

I have been applying the method of finite or infinite dimensional Lyapunov-Schmidt reduction in several different contexts, both in asymptotic regimes (when some parameter tends to 0) and also in non asymptotic regimes (when no parameter is present in the equations).

I have studied the following problems:

(1) Entire solutions for non-linear Schrödinger equation
(2) Singularly perturbed Neumann problem in bounded domain
(3) Chern-Simons equation and Toda system
(4) Magnetic Ginzburg-Landau equation and Chern-Simons-Higgs equation

2.1 Solutions without any symmetry for semilinear elliptic problems

In [9], I consider the classical non-linear stationary Schrödinger equation
\[
\Delta u - u + u^3 = 0 \text{ in } \mathbb{R}^2
\]  \hfill (1)

We develop tools to construct infinitely many entire solutions with a small group of symmetry and with finite energy, namely solutions u to (1) so that their energy
\[
\mathcal{E}(u) := \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) \, dx - \frac{1}{4} \int_{\mathbb{R}^2} u^4 \, dx,
\]
is finite.

Equations like (1), in dimension 2 or in higher dimensions, have been thoroughly studied over the last decades since they are ubiquitous in various models in mathematical physics or biology. For example, the study of standing waves (or solitary waves) for the nonlinear Klein-Gordon or Schrödinger equations reduces to (1).

The classical result of Gidas, Ni and Nirenberg [15] asserts that any finite energy, positive solution of (1) are all radial symmetric. So non-radial solutions must be sign-changing. As far as we know, up to today, all sign-changing solutions we know has some non-trivial group of symmetry. A natural question is the following :
Do all solutions of (1) have a nontrivial group of symmetry?

Surprisingly, the answer to this question is negative. In fact, we prove

**Theorem 2.1.** ([9]) There exist infinitely many solutions of (1) which have finite energy but whose group of symmetry reduces to the identity.

The proof of this result relies on an extension of the construction in [25]. The idea is to apply geometric constructions of N. Kapouleas on constant mean curvature surface in Euclidean three space to semilinear elliptic equations. We are able to obtain some general condition for the construction of such solutions, and are able to provide concrete examples of this construction.

### 2.2 On the Singularly Perturbed Neumann Problem

Another problem I consider is the following singularly perturbed Neumann problem:

\[
\begin{cases}
\varepsilon^2 \Delta u - u + u^p = 0 \text{ in } \Omega \\
u > 0 \text{ in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega,
\end{cases}
\]

where \( p \) satisfies \( 1 < p < +\infty \) for \( n = 2 \) and \( 1 < p < \frac{n+2}{n-2} \) for \( n \geq 3 \) and \( \Omega \) is bounded, smooth domain in \( \mathbb{R}^n \) with its unit outward normal \( \nu \). This is a classical problem in asymptotic analysis and non-linear partial differential equation, which has been widely studied in the last twenty years. A key property of this equation is that the solutions exhibit varies concentration phenomena.

One problem is the number of interior spike solutions to (2). Note that since \( p \) is subcritical, the solutions to (2) is uniformly bounded. Thus the energy bound for solutions of (2) is \( O(1) \). On the other hand, each spike contributes to at least \( O(\varepsilon^n) \) energy. This implies that the number of interior spikes can not exceed \( O(\varepsilon^{-n}) \). In [20], it is showed that there are at least \( \frac{\delta(n, p, \Omega)}{\varepsilon n \log \varepsilon} \) number of interior spikes. In [4], we improve the upper bound to \( \frac{\delta(n, p, \Omega)}{\varepsilon^n} \) which is optimal.

The key point of our proof is the use of “localized energy method” as in [20]. There are two main difficulties. First, the distance between spikes is assumed only to be \( O(\varepsilon) \). In the Liapunov-Schmidt reduction process, we have to prove that all the estimates are uniform with respect to the integer \( k \). Second, we have to detect the difference in the energy when spikes move to the boundary of the configuration space. In the second step, we perform a secondary Liapunov-Schmidt reduction to get a accumulated error estimate from step \( k \) to step \( k+1 \).

**Theorem 2.2.** ([4]) There exists an \( \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon < \varepsilon_0 \), and any positive integer \( k \) satisfying

\[
1 \leq k \leq \frac{\delta(\Omega, n, p)}{\varepsilon^n},
\]

where \( \delta(\Omega, n, p) \) is a constant depending on \( n, \Omega \) and \( p \) only, problem (2) has a solution \( u_\varepsilon \) that possesses exactly \( k \) local maximum points.
It has been conjectured that problem (2) possesses solutions which have m-dimensional concentration sets for $0 \leq m \leq n - 1$.

Boundary concentration solutions have been constructed, either concentrating on the whole boundary or minimal submanifold of the boundary. As far as we know, all boundary concentration sets are closed geodesics. A natural question is the following:

*Does problem (2) has solutions which concentrate on a broken segment of the boundary?*

In [1], we get an affirmative answer to this question. We construct solutions concentrating on a broken segment $\gamma$ of the boundary $\partial \Omega \subset \mathbb{R}^2$ if $\gamma$ satisfies the following condition:

\[(H_1). \text{ Let } \gamma = \gamma([0,b]) \text{ be the segment parametrized by arc length, and } H(p) \text{ be the curvature of } \partial \Omega \text{ at } p. \text{ Denote by}\]
\[
H'(\gamma(s)) = \frac{d}{ds}H(\gamma(s)), \quad H''(\gamma(s)) = \frac{d^2}{ds^2}H(\gamma(s)).
\]

Assume that $H''(\gamma(s)) \geq c_0 > 0$ for all $s \in [0,b]$, and $\int_0^b H'(\gamma(s))ds = 0$.

**Theorem 2.3.** Assume that $\gamma$ satisfies $(H_1)$, then there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, there exists boundary spike solutions to (2) concentrating on $\gamma$.

This is a new concentration phenomena. The motivation of our construction comes from the study of the constant mean curvature surface by Butscher and Mazzeo in [2], where they constructed CMC surface condensing to a geodesic segments by connecting large number ($O(1/r)$) of spheres of radius $r$ distributing along the geodesic segment. The main difficulty is the construction of balance approximate solutions, where we use a new ode method and consider the problem as a discretization of its continuum limiting ode system. Using this idea we can construct almost balance approximate solutions.

For the same problem, I also obtained solutions to (2) whose energy concentrate along curves *inside* the domain $\Omega$. In paper [2], we consider solutions concentrating on *interior straightline* intersecting with the boundary orthogonally and in [3], we construct the $Y$-shaped spike solution, i.e. solution concentrating at three interior lines which intersect at a same point.

Using the same method, in [5], we also get the existence of infinitely many spike solutions for the following non-linear Schrodinger equation:

\[
\begin{cases}
\Delta u - (1 + \delta V)u + f(u) = 0 \quad \text{in } \mathbb{R}^n \\
u \in H^1(\mathbb{R}^n)
\end{cases}
\]

where $V$ is a potential satisfying some decay condition but without any symmetry assumption:

\[
\begin{cases}
(H_1) \quad V(x) \to 0 \quad \text{as } |x| \to \infty, \\
(H_2) \quad \exists \quad 0 < \bar{\eta} < 1, \lim_{|x| \to \infty} V(x) e^{\bar{\eta}|x|} = +\infty, \\
(H_3) \quad V \text{ is continuous in } \mathbb{R}^n,
\end{cases}
\]

and $f(u)$ is a superlinear nonlinearity satisfying some nondegeneracy condition:

\[(f_1) \quad f : \mathbb{R} \to \mathbb{R} \text{ is of class } C^{1+\sigma} \text{ for some } 0 < \sigma \leq 1 \text{ and } f(u) = 0 \text{ for } u \leq 0, f'(0) = 0.\]
The equation
\[
\begin{cases}
\Delta w - w + f(w) = 0, \ w > 0 \text{ in } \mathbb{R}^n \\
w(0) = \max \{ w(y) \mid y \in \mathbb{R}^n \}, \ w \to 0 \text{ as } |y| \to \infty
\end{cases}
\]  
(6)
has a non-degenerate solution \( w \).

**Theorem 2.4.** ([5]) Let \( f \) satisfy assumptions (\( f_1 \)) - (\( f_2 \)) and the potential \( V \) satisfy assumptions (\( H1 \)) - (\( H3 \)). Then there exists a positive constant \( \delta_0 \), such that for \( 0 < \delta < \delta_0 \), problem (4) has infinitely many positive solutions.

### 2.3 Non-topological Solutions of the rank 2 Chern-Simons system

Another line of my research is the study of solutions for systems of non-linear elliptic differential equation. One problem I consider is the following Chern-Simons system:

\[
\begin{align*}
\Delta u_1 + \left( \sum_{i=1}^{2} K_{1i} e^{u_i} - \sum_{i=1}^{2} \sum_{j=1}^{2} e^{u_i} K_{1i} e^{u_j} K_{ij} \right) &= 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\
\Delta u_2 + \left( \sum_{i=1}^{2} K_{2i} e^{u_i} - \sum_{i=1}^{2} \sum_{j=1}^{2} e^{u_i} K_{2i} e^{u_j} K_{ij} \right) &= 4\pi \sum_{j=1}^{N_2} \delta_{q_j}
\end{align*}
\]  
(7)

where \( K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \) is the Cartan matrix of rank 2 of the Lie algebra \( \mathcal{G} \) (there are three Cartan matrix of rank 2: \( A_2, B_2, G_2 \)), \( \{ p_1, \ldots, p_{N_1} \} \) and \( \{ q_1, \ldots, q_{N_2} \} \) are given vortex points. This system is derived from the Non-Abelian gauge field theory, and relativistic Chern-Simons model to explain the physics of high critical temperature superconductivity. In geometry, solutions of Toda system is closely related to holomorphic curves in projective spaces. For example, the Toda system of type \( A_m \) can be derived from the classical Plücker formulas, and any holomorphic curve give rise to a solution \( u \) of the Toda system, which branch points corresponds to the singularities of \( u \). Conversely, we could integrate the Toda system, and any solution \( u \) gives rise to a holomorphic curve in \( \mathbb{CP}^n \) at least locally.

A long-standing open problem for (7) is the question of the existence of non-topological solutions. **In [6] and [7], we give the first mathematically rigorous proof for the existence of such a solution when the Cartan matrix \( K \) is of rank 2.**

We will view equation (7) as a small perturbation of the \( A_2, B_2 \) and \( G_2 \) Toda system with singular sources respectively. A key point we use is the complete classification and non-degeneracy of solutions to the \( SU(N+1) \) Toda system with singular sources which was obtained by Lin, Wei and Ye in [22].

**Theorem 2.5.** ([6], [7]) Let \( \{ p_j \}_{j=1}^{N_1}, \{ q_j \}_{j=1}^{N_2} \subset \mathbb{R}^2 \). If either

(a) \( N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j \);

or

(b) \( N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j \) and \( N_1, N_2 > 1, \ |N_1 - N_2| \neq 1 \), then there exists a non-topological solution \((u_1, u_2)\) of problem (7) when \( K \) is cartan matrix of rank 2.
As we mentioned before, a key point is to get the classification and non-degeneracy for the rank 2 Toda system. In [6], [8], we get the classification and non-degeneracy result and we use this non-degeneracy result to prove the existence of non-topological solutions. A key ingredient in the proof of classification of rank 2 Toda system with singular sources is the observation that Toda system with $B_2$ can be embedded into Toda system with $A_3$ and Toda system with $G_2$ can be embedded into Toda system $A_6$ under suitable group actions.

2.4 The non self-dual Chern-Simons-Higgs model

Another problem I consider is more related to physics. In [10], we consider the existence of non-radial symmetric solutions to the magnetic Chern-Simons-Higgs equation. The magnetic Chern-Simons-Higgs (CSH) equations are

\[-\Delta_A \psi + \lambda^2 (1 - |\psi|^2)(1 - 3|\psi|^2)\psi - \frac{\mu^2}{4} \frac{|\nabla \times A|^2}{|\psi|^4} \psi = 0\] (8)

\[\frac{\mu^2}{4} \nabla \times \left( \frac{\nabla \times A}{|\psi|^2} \right) + \text{Im}(\bar{\psi} \nabla A \psi) = 0\] (9)

for $\lambda, \mu > 0$ constants, where $\psi : \mathbb{R}^2 \to \mathbb{C}$ and $A : \mathbb{R}^2 \to \mathbb{R}^2$. $\nabla_A = \nabla - iA$ is the covariant gradient, and $\Delta_A = \nabla_A \cdot \nabla_A$. For a vector field $A$, $\nabla \times A$ is the scalar $\partial_1 A_2 - \partial_2 A_1$ and for scalar $\xi$, $\nabla \times \xi$ is the vector $(-\partial_2 \xi, \partial_1 \xi)$. The CSH equations arise from the problems in condensed matter physics such as high-temperature superconductivity and quantum and fractional Hall effect.

Unlike the self-dual case $\lambda = \frac{1}{\mu}$, where the CSH equation can be reduced to Bogomolnyi-type first order equations, and there has been rich mathematical development in this direction, less is known for the non self-dual case $\lambda \neq \frac{1}{\mu}$. In [10], we consider the existence of non-radial symmetric solutions whose symmetry group reduces to identity. Using the idea of dealing with the non-linear Schrödinger equation in Section 2.2, we proved the following existence result:

**Theorem 2.6.** There exists $\varepsilon_0 > 0$ small, and for fixed $\lambda > \frac{1}{\mu}$ with $\lambda - \frac{1}{\mu} < \varepsilon_0$, there exists infinitely many solutions of (8) and (9) which have finite energy but without any symmetry.

The building block we use is the $n$ vortex solutions:

\[\psi^{(n)}(x) = f_n(r)e^{in\theta} \quad \text{and} \quad A^{(n)}(x) = a_n(r)\nabla(n\theta)\] (10)

where $(f_n, a_n)$ is minimizer of the energy functional in the radial space:

\[G(f, a) = \frac{1}{2} \int_{\mathbb{R}^2} |f'|^2 + n^2(1 - a)^2 f^2 + \frac{\mu^2}{4} \frac{n^2}{r^2} (a')^2 + \lambda^2 f^2 (1 - f^2)^2 dx.\] (11)

The existence of the $n$ vortex solutions is known [14], but the potential term $\lambda^2 f^2 (1 - f^2)^2$ prevents one from getting convexity of the energy functional. Thus there is no uniqueness and non-degeneracy results for the minimizer.

Since we employ the idea in [9], where the Lyapunov-Schmidt reduction method is used. The key property is the non-degeneracy of the building blocks. Our new idea is to use the non-degeneracy of the topological solutions for the self-dual case, i.e. $\lambda = \frac{1}{\mu}$ and use perturbation argument to construct non-degenerate $n$ vortex solutions in the form (10) and using these vortex as our new building blocks to construct the solutions.
3 Research Plans

In what follows, I will discuss what I am doing now and going to do in the next few years.

3.1 The self-dual Chern-Simons system

I will continue the study on the Chern-Simons system. In more general, the self-dual system takes the form:

\[ \Delta u_i = \lambda \left( \sum_{j=1}^{n} \sum_{k=1}^{n} K_{ij} e^{u_j} K_{jk} e^{u_k} - \sum_{j=1}^{n} K_{ij} e^{u_j} \right) + 4\pi \sum_{s=1}^{N_i} \delta_{p_{is}}, \]

for \( i = 1, \ldots, n \), or equivalently, \( u = (u_1, \ldots, u_n)^t \),

\[ \Delta u = \lambda [KUK - K]U + 4\pi \sum_{s=1}^{N_i} \delta_{p_{is}}, \]

where \( U = \text{diag}\{e^{u_1}, \ldots, e^{u_n}\}, \ U = (e^{u_1}, \ldots, e^{u_n})^t, \) and \( K \) is the \( n \times n \) Cartan matrix. It is easy to check that if \( u(x) \) is a solution of (13) such that \( u(x) \to \) a constant vector \( v_0 \) as \( |x| \to \infty \), then

\[ v_{0,i} = \ln \left( \sum_{j=1}^{n} (K^{-1})_{ij} \right), \]

where \((K^{-1})_{ij}\) is the inverse of \( K \). Thus we always assume that \( K \) satisfies

(H). \( K \) is invertible and \( \sum_{j=1}^{n} (K^{-1})_{ij} > 0 \) for all \( i \).

In the literature, under the assumption (H), a solution \( u \) is called a topological solution if

\[ u_i(x) \to \ln \left( \sum_{j=1}^{n} (K^{-1})_{ij} \right), \] as \( |x| \to \infty, \)

and is called a non-topological solution if

\[ u_i(x) \to -\infty \] as \( |x| \to \infty. \)

The above Chern-Simons system has been known for twenty years, but very few mathematical results were known after the pioneering work of Yisong Yang ([28]). Only recently it got some attention from mathematicians. Equation (13) is a complicated system, and can be written as a variational form:

\[ J(\phi) = \frac{1}{2} \sum_{i,j} \int_{\mathbb{R}^2} b_{ij} \nabla \phi_i \cdot \nabla \phi_j dx + \frac{1}{2} \int_{\mathbb{R}^2} (\sum_{i,j} K_{ij} e^{u_1} e^{u_2} + \phi_i + \phi_j \]

\[ \sum_{i} e^{u_i \phi_i} dx + \sum_{i} \int_{\mathbb{R}^2} h_i \phi_i u dx \]

if \( K \) is symmetric, where \( K^{-1} = (b_{ij}), h_i = \sum_j b_{ij} g_j, \) and

\[ \Delta u_{i,0} = 4\pi \sum_{s=1}^{N_i} \delta_{p_{is}} - g_i, \]

\( \gamma_i = \sum_{s=1}^{n_i} \frac{4}{\left( 1 + |x - p_{is}|^2 \right)^2}. \)
The first problem is on the existence of topological solutions. Suppose $K$ satisfies (H), dose a topological solution exist for any configuration $p_{ij} \in \mathbb{R}^2$. Yang Yisong gave a affirmative answer to this question for a special class of $K$:

(S) $K = PS$, where $S$ is positive definite and $P$ is a diagonal matrix with positive diagonal entries.

Note that the $SU(n+1)$ satisfies the above condition (S). Yang prove this result by applying the Moser-Trudinger inequality to obtain a minimizer of $J$. Since $S$ is positive definite, the quadratic form is coercive. But this method fails if (S) is not satisfied. In [21], Lin-Ponce-Yang consider the Chern-Simons system when

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

In this case the equation becomes

$$\begin{cases} 
\Delta u_1 = \lambda e^{u_2}(e^{u_1} - 1) + 4\pi \sum_{s=1}^{N_1} \delta_{p_{1s}}, \\
\Delta u_2 = \lambda e^{u_1}(e^{u_2} - 1) + 4\pi \sum_{s=1}^{N_2} \delta_{p_{2s}}.
\end{cases} \quad (15)$$

In this case, $K$ does not satisfy (S) and Yang’s method can not apply. They used an iterative scheme and Moser-Trudinger inequality to prove the existence of topological solutions. It has been mentioned later by the authors that the monotone scheme can be used. Thus there exists a maximal solution to (15), which is also a topological solution. Moreover, if there is only one singular data, then by the moving plane method, one know that all the topological solutions are radial and it has been proven in [12] that all radial topological solutions are unique. But for topological solutions of (13) for general configurations, the uniqueness problem is totally open even for the $2 \times 2$ case.

Based on Lin-Ponce-Yang’s result, it is possible to consider the existence of topological problem for $2 \times 2$ matrix $K$ which has the following form:

$$K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (16)$$

where $a, b, c, d > 0$ and $b, c > \max a, d$. In this case, $K$ satisfies (H) but is not positive definite. So it does not satisfy (S), Yang’s method can not use.

One possible method is to use the degree method, and deform the problem to the case with $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. So the key point is to prove the uniform boundedness of the solution under such deformation and to get the degree of solutions for $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For non-topological solutions of (13), we have mentioned before that our result is only the first step to the full understanding of the problem. Recently, Huang and Lin ([17], [18]) studied the structure of radial solutions for the rank 2 Chern-Simons system. Among other things, they proved the following classification result:

**Theorem A** Suppose $(u(r), v(r))$ is an entire radial solution to the $SU(3)$ Chern-Simons system. One of the following holds:

(i) $\lim_{r \to \infty} (u, v) = (0, 0)$,
(ii) \( \lim_{r \to \infty} (u, v) = (-\infty, -\infty) \), and \( e^u, e^v \in L^1(\mathbb{R}^2) \),

(iii) \( \lim_{r \to \infty} (u, v) = (\log \frac{1}{2}, -\infty) \) or \( (-\infty, \log \frac{1}{2}) \) which is called a mixed type solution. Furthermore, \( e^u \in L^1(\mathbb{R}^2) \) if \( u \to -\infty \) as \( r \to \infty \), \( e^v \in L^1(\mathbb{R}^2) \) if \( v \to -\infty \) as \( r \to \infty \).

For the asymptotic behaviour of these solution, they proved that if \( (u_1, u_2) \) is a radially symmetric non-topological solution to \( Su(3) \) Chern-Simons system with all vortices at the origin, then

\[
 u_1(r) = -2\alpha_1 \log r + O(1), \quad u_2(r) = -2\alpha_2 \log r + O(1) \tag{17}
\]

at infinity for some \( \alpha_1, \alpha_2 > 1 \). Furthermore,

\[
 J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1),
\]

where \( J(x, y) \) is the quadratic form associated to \( A_2^{-1} \).

For the existence of radially symmetric non-topological solutions, Choe, Kim and the second author [13] proved the following result:

**Theorem B** If \( (\alpha_1, \alpha_2) \) defined in (17) satisfies

\[
 -2N_1 - N_2 - 3 < \alpha_2 - \alpha_1 < 2N_2 + N_1 + 3,
2\alpha_1 + \alpha_2 > N_1 + 2N_2 + 6 \quad \text{and} \quad \alpha_1 + 2\alpha_2 > 2N_1 + N_2 + 6,
\]

then the \( A_2 \) Chern-Simons system has a radially symmetric solution \( (u_1, u_2) \) subject to the boundary condition (17).

For the radial solutions, one natural question is whether they are unique or not. So it is important to know the non-degeneracy of the linearized equations.

Another question is that whether all the three types of solutions exist. In particular, it is interesting the study the existence of mixed type solutions in the entire space, and the corresponding bubbling solutions in torus.

For more general Cartan matrix other than the rank 2 matrix, the classification of entire radial solutions is completely open.

### 3.2 The non self-dual Chern-Simons-Higgs equation

As I mentioned in Section 2.4, we have extended the result in [10] to other mathematical models like the Magnetic Chern-Simons-Higgs equation and Magnetic Ginzburg-Landau equation. In our proof, one key point is to know the non-degeneracy of the building blocks. For magnetic Ginzburg-Landau equation, we have everything we need in hand, but for the magnetic Chern-Simons-Higgs equation, The above equation can be viewed as the Euler-Lagrange equations of the following Chern-Simons-Higgs energy:

\[
 E(\psi, A) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \psi|^2 + \frac{\mu^2}{4} \frac{|\nabla \times A|^2}{|\psi|^2} + \lambda^2 |\psi|^2 (1 - |\psi|^2)^2. \tag{18}
\]

When \( \mu = \frac{1}{\lambda} \) (which is the self-dual case), the minimizers of the CSH energy satisfy a simpler system of first order PDE and has been extensively studied. In [14], Chen and Spirn initiate the study of the CSH energy outside the self-dual regime. In this paper, they studied the radially symmetric fields of the form:

\[
 \psi^{(n)} = f^{(n)}(r)e^{in\theta}, \quad A^{(n)} = a_n(r) \nabla(n\theta). \tag{19}
\]
They proved that for any \( n \), there exists solutions of the form (19) to (8) and (9), and the radial functions \((f^{(n)}_n, a_n)\) minimize the radial energy functional (18).

Although the radial solution minimizes the radial functional, but one can not prove that it is non-degenerate. So one problem is to prove the non-degeneracy of the radial solutions and after that, it is natural to study the stability of the full CSH energy (18) as was done by Gustafson and Sigal [16] for the magnetic Ginzburg-Landau equation, but the Hessian will be more complicated than the Hessian of the Ginzburg-Landau energy.

3.3 The constant mean curvature surfaces

As I have mentioned in Section 2.2, we constructed solutions to the singularly perturbed Neumann problem concentrating on boundary segments. The motivation of our construction comes from the study of the constant mean curvature surface. In [11], Butscher and Mazzeo constructed CMC surface condensing to a geodesic segments by connecting large number \((O(\frac{1}{r}))\) of spheres of radius \( r \) distributing along the geodesic segment. In their paper, they require the symmetry condition on the geodesic segment. In our construction, we do not need the symmetry condition for the segment. We believe that our idea can be used to construct CMC surface condensing to geodesic segments without the symmetry condition.

Another direction is to study the extension of the results in Section 2.2 and 2.3 for the second order elliptic equations to the fractional Laplacian problem.

References


