Approximating the Arctan Function

With many asides

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Builder's Approximation of Arctan

Letter received by UBC Professor, June 1992

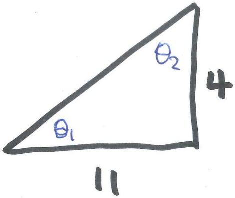
I have a small problem which has been plaguing me and my fellow builders for some time now.

What we are seeking is a mathematical equation to determine rafter angles. I came up with what I thought was a good formula but it does not seem to quite work. As you can see from the diagram, I began with a roof with a pitch of $4/11$. Assuming of course that the right angle is 90°, this means the other two angles must add up to 90°. I added 4 and 11 together obtaining 15. I then divided 90° by 15 giving me 6°. From there I multiplied $6^\circ \times 11$ to give me 66° as one angle and $6^\circ \times 4$ to give me 24° as the other angle.

Now, although we thought we were on the right track here, we found the rafters were always out by a few degrees. We have not ruled out the possibility that we made a mistake in the measurement somewhere but before we go through the whole process again, perhaps you could tell us if our equation is correct or not. If it is not, please put us on to the correct equation (try to keep it fairly simple as we have not been in school for quite a few years and our math skills are a bit rusty). Thank you.

Builder's Approximation of Arctan

Diagram



Builder's Approximation

$\theta_1 \approx B\left(\frac{4}{11}\right) = 24^\circ$

$\theta_2 \approx B\left(\frac{11}{4}\right) = 66^\circ$

↑ approximation of arctan.

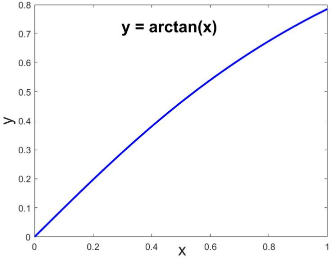
Exact.

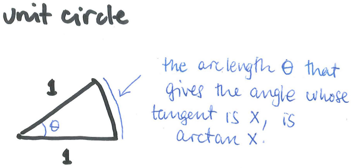
↓

$\theta_1 = \arctan\left(\frac{4}{11}\right) = 19.9831\ldots$

$\theta_2 = \arctan\left(\frac{11}{4}\right) = 70.0169\ldots$

Arctan Function

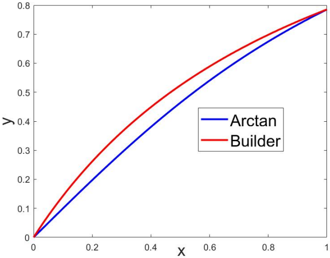




- $\arctan(x) + \arctan(1/x) = \pi/2$
- $\arctan(x)$ can also be written $\tan^{-1}(x)$ but that can be confusing
- Be careful when the resulting angle is not in the first quadrant

Builder's Approximation

$$B(x) = \frac{\pi}{2}\frac{x}{1+x}$$



- $B(x)$ preserves the property $B(x) + B(1/x) = \pi/2$.
- It is exact at values 0 and 1.
- It can be evaluated using only basic arithmetic operations (addition, subtraction, multiplication, and division).
- It is a low order Padé approximation.

Builders Need a Scientific Calculator

Or a table

TABLE III

12°

'	sin	tan	cot	cos	
0	.20791	.21256	4.7046	.97815	60
1	820	286	.6979	809	59
2	848	316	.6912	803	58
3	877	347	.6845	797	57
4	905	377	.6779	791	56
5	.20933	.21408	4.6712	.97784	55
6	962	438	.6646	778	54
7	.20990	469	.6580	772	53
8	.21019	499	.6514	766	52
9	047	529	.6448	760	51
10	.21076	.21560	4.6382	.97754	50
11	104	590	.6317	748	49
12	132	621	.6252	742	48
13	161	651	.6187	735	47
14	189	682	.6122	729	46
15	.21218	.21712	4.6057	.97723	45
16	246	743	.5993	717	44
17	275	773	.5928	711	43
18	303	804	.5864	705	42
19	331	834	.5800	698	41
20	.21360	.21864	4.5736	.97692	40
21	388	895	.5673	686	39
22	417	925	.5609	680	38
23	445	956	.5546	673	37
24	474	.21986	.5483	667	36
25	.21502	.22017	4.5420	.97661	35
26	530	047	.5357	655	34
27	559	078	.5294	648	33
28	587	108	.5232	642	32
29	616	139	.5169	636	31
30	.21644	.22169	4.5107	.97630	30
31	672	200	.5045	623	29
32	701	231	.4983	617	28
33	729	261	.4922	611	27
34	758	292	.4860	604	26
35	.21786	.22322	4.4799	.97598	25
36	814	353	.4737	592	24
37	843	383	.4676	585	23
38	871	414	.4615	579	22
39	899	444	.4555	573	21
40	.21928	.22475	4.4494	.97566	20
41	956	505	.4434	560	19
42	.21985	536	.4373	553	18
43	.22013	567	.4313	547	17
44	041	597	.4253	541	16
45	.22070	.22628	4.4194	.97534	15
46	098	658	.4134	528	14
47	126	689	.4075	521	13
48	155	719	.4015	515	12
49	183	750	.3956	508	11
50	.22212	.22781	4.3897	.97502	10
51	240	811	.3838	496	9
52	268	842	.3779	489	8
53	297	872	.3721	483	7
54	325	903	.3662	476	6
55	.22353	.22934	4.3604	.97470	5
56	382	964	.3546	463	4
57	410	.22995	.3488	457	3
58	438	.23026	.3430	450	2
59	467	056	.3372	444	1
60	.22495	.23087	4.3315	.97437	0
	cos	cot	tan	sin	'

77°

13°

'	sin	tan	cot	cos	
0	.22495	.23087	4.3315	.97437	60
1	523	117	.3257	430	59
2	552	148	.3200	424	58
3	580	179	.3143	417	57
4	608	209	.3086	411	56
5	.22637	.23240	4.3029	.97404	55
6	665	271	.2972	398	54
7	693	301	.2916	391	53
8	722	332	.2859	384	52
9	750	363	.2803	378	51
10	.22778	.23393	4.2747	.97371	50
11	807	424	.2691	365	49
12	835	455	.2635	358	48
13	863	485	.2580	351	47
14	892	516	.2524	345	46
15	.22920	.23547	4.2468	.97338	45
16	948	578	.2413	331	44
17	.22977	608	.2358	325	43
18	.23005	639	.2303	318	42
19	033	670	.2248	311	41
20	.23062	.23700	4.2193	.97304	40
21	090	731	.2139	298	39
22	118	762	.2084	291	38
23	146	793	.2030	284	37
24	175	823	.1976	278	36
25	.23203	.23854	4.1922	.97271	35
26	231	885	.1868	264	34
27	260	916	.1814	257	33
28	288	946	.1760	251	32
29	316	.23977	.1706	244	31
30	.23345	.24008	4.1653	.97237	30
31	373	039	.1600	230	29
32	401	069	.1547	223	28
33	429	100	.1493	217	27
34	458	131	.1441	210	26
35	.23486	.24162	4.1388	.97203	25
36	514	193	.1335	196	24
37	542	223	.1282	189	23
38	571	254	.1230	182	22
39	599	285	.1178	176	21
40	.23627	.24316	4.1126	.97169	20
41	656	347	.1074	162	19
42	684	377	.1022	155	18
43	712	408	.0970	148	17
44	740	439	.0918	141	16
45	.23769	.24470	4.0867	.97134	15
46	797	501	.0815	127	14
47	825	532	.0764	120	13
48	853	562	.0713	113	12
49	882	593	.0662	106	11
50	.23910	.24624	4.0611	.97100	10
51	938	655	.0560	093	9
52	966	686	.0509	086	8
53	.23995	717	.0459	079	7
54	.24023	747	.0408	072	6
55	.24051	.24778	4.0358	.97065	5
56	079	809	.0308	058	4
57	108	840	.0257	051	3
58	136	871	.0207	044	2
59	164	902	.0158	037	1
60	.24192	.24933	4.0108	.97030	0
	cos	cot	tan	sin	'

76°

14°

TABLE III

'	sin	tan	cot	cos	
0	.24192	.24933	4.0108	.97030	60
1	220	964	.0058	023	59
2	249	.24995	4.0009	015	58
3	277	.25026	3.9959	008	57
4	305	056	.9910	.97001	56
5	.24333	.25087	3.9861	.96994	55
6	362	118	.9812	987	54
7	390	149	.9763	980	53
8	418	180	.9714	973	52
9	446	211	.9665	966	51
10	.24474	.25242	3.9617	.96959	50
11	503	273	.9568	952	49
12	531	304	.9520	945	48
13	559	335	.9471	937	47
14	587	366	.9423	930	46
15	.24615	.25397	3.9375	.96923	45
16	644	428	.9327	916	44
17	672	459	.9279	909	43
18	700	490	.9232	902	42
19	728	521	.9184	894	41
20	.24756	.25552	3.9136	.96887	40
21	784	583	.9089	880	39
22	813	614	.9042	873	38
23	841	645	.8995	866	37
24	869	676	.8947	858	36
25	.24897	.25707	3.8900	.96851	35
26	925	738	.8854	844	34
27	954	769	.8807	837	33
28	.24982	800	.8760	829	32
29	.25010	831	.8714	822	31
30	.25038	.25862	3.8667	.96815	30
31	066	893	.8621	807	29
32	094	924	.8575	800	28
33	122	955	.8528	793	27
34	151	.25986	.8482	786	26
35	.25179	.26017	3.8436	.96778	25
36	207	048	.8391	771	24
37	235	079	.8345	764	23
38	263	110	.8299	756	22
39	291	141	.8254	749	21
40	.25320	.26172	3.8208	.96742	20
41	348	203	.8163	734	19
42	376	235	.8118	727	18
43	404	266	.8073	719	17
44	432	297	.8028	712	16
45	.25460	.26328	3.7983	.96705	15
46	488	359	.7938	697	14
47	516	390	.7893	690	13
48	545	421	.7848	682	12
49	573	452	.7804	675	11
50	.25601	.26483	3.7760	.96667	10
51	629	515	.7715	660	9
52	657	546	.7671	653	8
53	685	577	.7627	645	7
54	713	608	.7583	638	6
55	.25741	.26639	3.7539	.96630	5
56	769	670	.7495	623	4
57	798	701	.7451	615	3
58	826	733	.7408	608	2
59	854	764	.7364	600	1
60	.25882	.26795	3.7321	.96593	0
	cos	cot	tan	sin	'

75°

15°

'	sin	tan	cot	cos	
0	.25882	.26795	3.7321	.96593	60
1	910	826	.7277	585	59
2	938	857	.7234	578	58
3	966	888	.7191	570	57
4	.25994	920	.7148	562	56
5	.26022	.26951	3.7105	.96555	55
6	050	.26982	.7062	547	54
7	079	.27013	.7019	540	53
8	107	044	.6976	532	52
9	135	076	.6933	524	51
10	.26163	.27107	3.6891	.96517	50
11	191	138	.6848	509	49
12	219	169	.6806	502	48
13	247	201	.6764	494	47
14	275	232	.6722	486	46
15	.26303	.27263	3.6680	.96479	45
16	331	294	.6638	471	44
17	359	326	.6596	463	43
18	387	357	.6554	456	42
19	415	388	.6512	448	41
20	.26443	.27419	3.6470	.96440	40
21	471	451	.6429	433	39
22	500	482	.6387	425	38
23	528	513	.6346	417	37
24	556	545	.6305	410	36
25	.26584	.27576	3.6264	.96402	35
26	612	607	.6222	394	34
27	640	638	.6181	386	33
28	668	670	.6140	379	32
29	696	701	.6100	371	31
30	.26724	.27732	3.6059	.96363	30
31	752	764	.6018	355	29
32	780	795	.5978	347	28
33	808	826	.5937	340	27
34	836	858	.5897	332	26
35	.26864	.27889	3.5856	.96324	25
36	892	921	.5816	316	24
37	920	952	.5776	308	23
38	948	.27983	.5736	301	22
39	.26976	.28015	.5696	293	21
40	.27004	.28046	3.5656	.96285	20
41	032	077	.5616	277	19
42	060	109	.5576	269	18
43	088	140	.5536	261	17
44	116	172	.5497	253	16
45	.27144	.28203	3.5457	.96246	15
46	172	234	.5418	238	14
47	200	266	.5379	230	13
48	228	297	.5339	222	12
49	256	329	.5300	214	11
50	.27284	.28360	3.5261	.96206	10
51	312	391	.5222	198	9
52	340	423	.5183	190	8
53	368	454	.5144	182	7
54	396	486	.5105	174	6
55	.27424	.28517	3.5067	.96166	5
56	452	549	.5028	158	4
57	480	580	.4989	150	3
58	508	612	.4951	142	2
59	536	643	.4912	134	1
60	.27564	.28675	3.4874	.96126	0
	cos	cot	tan	sin	'

74°

How to Make the Table of Tangent Values?

Without a scientific calculator I

- $\tan 30^\circ = 1/\sqrt{3}$
- Trigonometric identity

$$\tan 2a = \frac{2\tan a}{1-\tan^2 a}$$

- So if $y = \tan 15^\circ$ ($\tan \pi/12$), we know that

$$\frac{1}{\sqrt{3}} = \frac{2y}{1-y^2}$$

- y solves the quadratic

$$y^2 + 2\sqrt{3}y - 1 = 0$$

  and $y = 2 - \sqrt{3} \approx 0.2679$.
- We can work out $\tan(30^\circ/2^n)$ for any n.

How to Make the Table of Tangent Values?

Without a scientific calculator II

- We can work out $\tan(30^\circ/2^n)$ for any n.
- Trigonometric identity

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

- We can work out $\tan(30^\circ p/2^n)$ for any integers p and n.
- (We can accurately interpolate desired values)
- This would let us complete the table.
- Be careful with the accuracy of intermediate computations

Aside: Computing Square Roots With Basic Arithmetic

Babylonian square root formula

- To find $x = \sqrt{A}$
- Iterative formula

$$x_{n+1} = \frac{1}{2}(x_n + \frac{A}{x^n})$$

  (the average of two values, one bigger than $\sqrt{A}$ and one smaller).
- The iterates x_n get closer and closer to $\sqrt{A}$ as n increases. We write

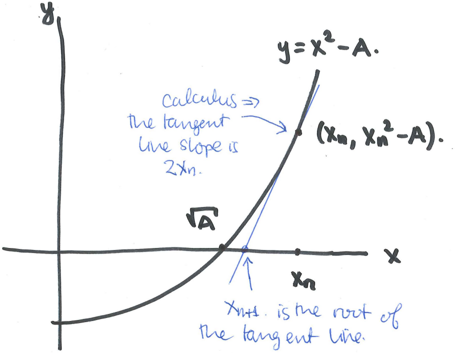
$$\lim_{n\to\infty} x_n = \sqrt{A}$$

- The convergence is quite rapid as shown below with $A = 2$ and $x_0 = 2$:

$$2 \to 1.5 \to 1.4167 \to 1.4142$$

Aside: Computing Square Roots With Basic Arithmetic

Babylonian square root formula is Newton's Method



- Tangent line (point–slope formula):

$$y = x_n^2 - A + 2x_n(x - x_n)$$

- Set $y = 0$ and solve for $x = x_{n+1}$:

$$x_{n+1} = x_n - \frac{x_n^2 - A}{2x_n} = \frac{1}{2}(x_n + A/x_n)$$

Aside: Computing Square Roots

- There is a systematic way to compute squate roots quickly on an abacus.
- You can use Log tables to compute square roots

$$\sqrt{x} = 10^{\frac{1}{2}\log_{10} x}$$

- You can use slide rules to compute square roots (they are built with a logarithmic scale).
- Log tables and slide rules give a quick way to multiply multi-digit numbers

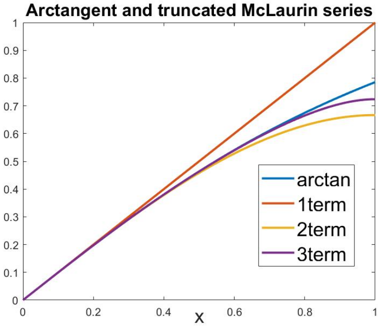
$$xy = 10^{\log_{10} x + \log_{10} y}$$

Approximating the Arctan Function for $x \in [0, 1]$

Use truncated McLaurin series I

$$\arctan x = x - x^3/3 + x^5/5 - x^7/7 + \ldots$$

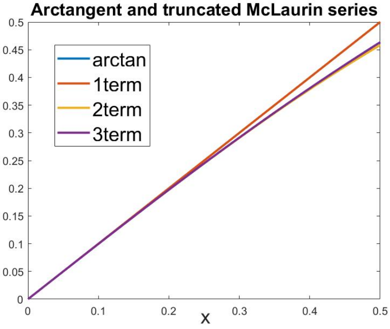
- The infinite series converges for $x \leq 1$.
- The truncated series involves only basic arithmetic operations.



Approximating the Arctan Function for $x \in [0, 1]$

Use truncated McLaurin series II

- The accuracy for $x \leq 1/2$ is much better than for x near 1.
- With 4 terms, the truncated McLaurin series has a relative error of less than 0.0001 for $x \leq 1/2$.



Approximating the Arctan Function for $x \in [0, 1]$

Use truncated McLaurin series III

- Still need an accurate formula for $x \in (1/2, 1]$
- if $x \in (1/2, 1]$,
$$\sqrt{\frac{1}{x^2}+1}-\frac{1}{x}$$
is in $(0, 1/2)$.
- Use the identity
$$\arctan x = 2\arctan\left(\sqrt{\frac{1}{x^2}+1}-\frac{1}{x}\right)$$
for the win.
- These are most of the ingredients for the implementation of arctangent on older floating point units. Current implementations are proprietary.