# Asymptotic Error Analysis

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# Overview of the Talk

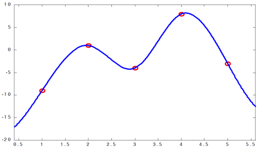
- Errors from computational methods using regular grids to compute smooth solutions have additional structure
- This structure can
  - allow Richardson Extrapolation
  - lead to super-convergence
  - guide the implementation of boundary conditions
  - help in the analysis of methods for non-linear problems
- Numerical artifacts (non-standard errors) can be present
- The process of finding the structure and order of errors can be called Asymptotic Error Analysis. Needs smooth solutions and regular grids.

# Interesting Facts

## Following Joshua's Introduction

- Richardson extrapolation of the Trapezoidal Rule is Simpson's Rule
- Trapezoidal and Midpoint Rules are spectrally accurate for integrals of periodic functions over their period

# Cubic Splines




- Given smooth $f(x)$ on [0,1], spacing $h = 1/N$, and data $a_i = f(ih)$ for $i = 0, \ldots N$ the standard cubic spline fit is a $C_1$ piecewise cubic interpolation.
- Cubic interpolation on each sub-interval for given values and second derivative values $c_i$ at the end points is fourth order accurate.
- If the second derivative values are only accurate to second order, the cubic approximation is still fourth order accurate.
- For $C_1$ continuity,

$$c_{i-1} + 4c_i + c_{i+1} = \frac{6}{h^2}(a_{i+1} - 2a_i + a_{i-1})$$

# Cubic Splines - Periodic Analysis

$$c_{i-1} + 4c_i + c_{i+1} = \frac{6}{h^2}(a_{i+1} - 2a_i + a_{i-1})$$

In this case, $c$ has a regular asymptotic error expansion

$$c = f'' + h^2(\frac{1}{12} - \frac{1}{6})f'''' + \dots$$

(the fact that $c_{i-1} + c_{i+1} = 2c_i + h^2 c'' + \dots$ is used). Since the $c$'s are second order accurate, the cubic spline approximation is fourth order accurate.

Notes:

- The earliest convergence proof for splines is in this equally spaced, periodic setting Ahlberg and Nilson, "Convergence properties of the spline fit", J. SIAM, 1963
- Lucas, "Asymptotic expansions for interpolating periodic splines," SINUM, 1982.

# Cubic Splines - Non-Periodic Case

$$c_{i-1} + 4c_i + c_{i+1} = \frac{6}{h^2}(a_{i+1} - 2a_i + a_{i-1})$$

In the non-periodic case, additional conditions are needed for the end values $c_0$ and $c_N$:

natural: $c_0 = 0, \ O(1)$

derivative: $2c_0 + c_1 = \frac{6}{h^2}(a_1 - a_0) - \frac{3}{h}f'(0), \ O(h^2)$

not a knot: $c_0 - 2c_1 + c_2 = 0, \ O(h^2)$

First convergence proof for "derivative" conditions Birkhoff and DeBoor, "Error Bounds for Spline Interpolation", J Math and Mech, 1964.

# Cubic Splines - Numerical Boundary Layer

$$c_{i-1} + 4c_i + c_{i+1} = \frac{6}{h^2}(a_{i+1} - 2a_i + a_{i-1})$$

No regular error can match the natural boundary condition $c_0 = 0$. However, note that

$$1 + 4\kappa + \kappa^2 = 0$$

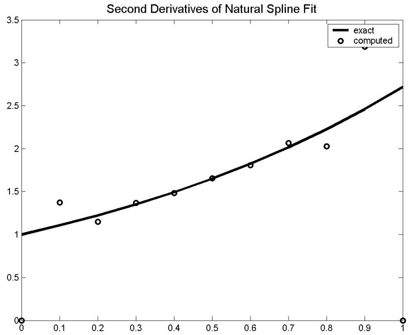
has a root $\kappa \approx -0.268$.

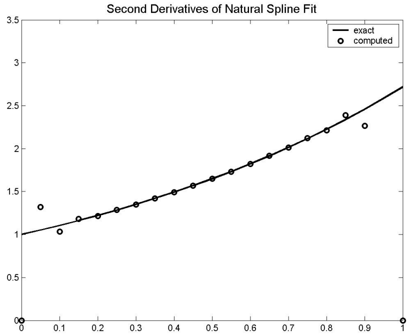
Error Expansion:

$$c_i = f''(ih) - h^2 \frac{1}{6} f''''(ih) - f''(0)\kappa^i \ldots$$

The new term is a *numerical boundary layer*. In this case, the spline fit will be second order near the ends of the interval and fourth order in the interior. Reference?

# Cubic Splines - Computation



# 1D Boundary Value Problem

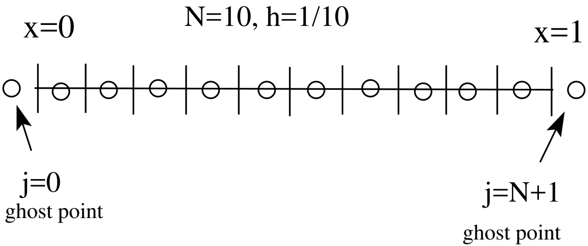
Simple boundary value problem for $u(x)$:

$$u'' - u = f \quad \text{with } u(0) = 0 \text{ and } u(1) = 0$$

with $f$ given and smooth.

Theory: Unique solution $u \in C^{k+2}$ for every $f \in C^k$.

- $N$ subintervals, spacing $h = 1/N$.
- Cell-Centred Finite Difference approximations $U_j \approx u((j - 1/2)h, j = 0 \ldots N + 1$.



# Uniform Grid

## Scheme

$$u'' - u = f \quad \text{with } u(0) = 0 \text{ and } u(1) = 0$$

- Finite Difference approximation for interior grid points

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} - U_j = f(jh)$$

truncation error $h^2 u''''(jh)/12 + O(h^4)$.

- Linear Interpolation of the boundary conditions

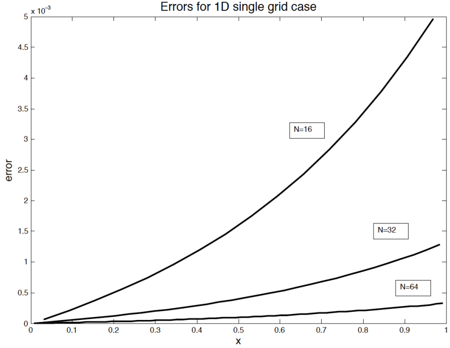
$$\frac{U_0 + U_1}{2} = 0$$

truncation error $h^2 u''(0)/8 + O(h^4)$.

Lax Equivalence Theorem: A stable, consistent scheme converges with the order of its truncation error.

# Uniform Grid

## Computational Results




Note that: the computed $U = u + h^2 u^{(2)} + O(h^4)$ with $u^{(2)}$ a smooth function of $x$. This is an asymptotic error expansion for $U$ with only regular terms (no artifacts).

# Uniform Grid

## Asymptotic error expansion

$$U = u + h^2 v(x) + O(h^4)$$

$$\begin{aligned} \frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} - U_j &= f(jh) + h^2 u''''(jh)/12 + O(h^4) \\ \frac{U_0 + U_1}{2} &= h^2 u''(0)/8 + O(h^4) \end{aligned}$$

Match terms at $O(h^2)$:

$$v'' - v = u''''/12 \quad \text{with } v(0) = u''(0)/8 \text{ and } v(1) = u''(1)/8$$

Asymptotic error term solves the original DE but forced by the truncation error.

# Uniform Grid

## Asymptotic error expansion discussion

$$U = u + h^2 v(x) + O(h^4)$$

- $v$ is just a theoretical tool, never computed.
- Justifies full order convergence of derivative approximations (super-convergence):

$$(U_{j+1} - U_{j-1})/(2h) = u_x(jh) + O(h^2)$$

- Justifes full order convergence of derivatives with parameters.
- Tool for theoretical analysis of nonlinear problems.

# Uniform Grid

## Be careful on interpreting BC accuracy

At the boundary we have

$$\frac{U_0 - 2U_1 + U_2}{h^2} - U_j = f(0) \quad \text{and} \quad \frac{U_0 + U_1}{2} = 0$$

These can be combined to give

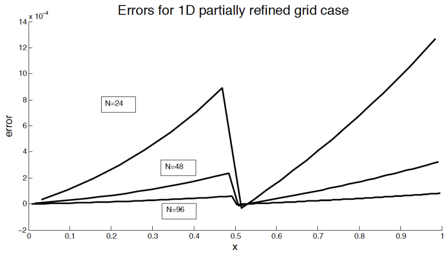
$$\frac{-3U_1 + U_2}{h^2} - U_j = f(0)$$

which is not consistent (errors do not $\to 0$ as $h \to 0$).

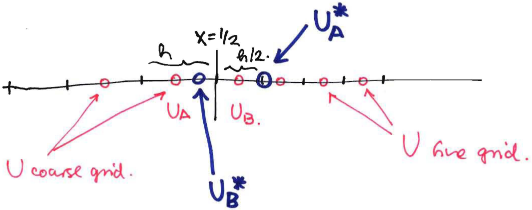
- Interpret BC accuracy in approximations of the original BCs
- Useful idea for implementing unusual BCs
- Higher order wide stencils introduce numerical boundary layers

# 1D Partially Refined Grid

- Refine the grid in the right half of the interval by a factor of 2.
- Ghost points at the refinement interface are related to grid values by linear interpolation/extrapolation.
- Second order convergence is seen in the solution.
- The computed $U$ has a piecewise regular error expansion.



# 1D Partially Refined Grid Analysis



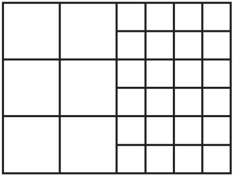

- Linear interpolation $U_B^* = \frac{2}{3}U_A + \frac{1}{3}U_B$
- Linear extrapolation $U_A^* = -\frac{1}{3}U_A + \frac{4}{3}U_B$
- Determine the accuracy at which the "interface" conditions $[u] = 0$ and $[u'] = 0$ are approximated.
- The conditions above can be rewritten as

$$\begin{aligned}(U_A + U_A^*)/2 &= (U_B + U_B^*)/2 \\ (U_A^* - U_A)/h &= (U_B - U_B^*)/(h/2)\end{aligned}$$

so are second order approximations of the interface conditions.

# Idealized Piecewise Regular Grid

Consider the idealized piecewise regular grid in 2D:



- At the interface, ghost points are introduced, related to grid points by linear extrapolation.
- Coarse grid has regular error $U_{coarse} = u + h^2 e_{coarse} + \ldots$
- Fine grid has regular error and an artifact

$$U_{fine} = u + h^2 e_{fine} + h^2 \frac{u_{xy}(0,y)}{8(1-\kappa)}(-1)^j \kappa^i + \cdots$$

- Artifact causes loss of convergence in $D_{2,y}U$ and $D_{2,x}U$ *on the fine grid side at the interface.*

# 2D Stokes Equations

## Simplest Framework

- Unknowns are velocities $\mathbf{u}(x, y, t) = (u, v)$ and pressure $p$.
- Momentum balance $\mathbf{u}_t = \Delta \mathbf{u} - \nabla p + \mathbf{f}$
- Incompressibility $\nabla \cdot \mathbf{u} = 0$
- The action of the pressure is to *project* the RHS of the momentum equations onto the space of divergence free fields with zero normal boundary values.
- Take $\mathbf{f}$ of the form $\mathbf{f}(x)e^{i(\omega t+\alpha y)}$ and look for solutions

$$\mathbf{u}(x)e^{it+iy}$$
$$p(x)e^{it+iy}$$

with $\mathbf{u} = 0$ at $x = 0, 1$.

# 2D Stokes Equations

## Coupled BDF2

$$\begin{aligned} u'' - (1+i)u - p' &= f_1 \\ v'' - (1+i)v - ip &= f_2 \\ u' + iv &= g \end{aligned}$$

- Coupled BVPs for $u$, $v$, and $p$.
- BDF2 time stepping approximates $u_t$ by

$$\frac{1}{k}\left(\frac{3}{2}U^n - 2U^{n-1} + \frac{1}{2}U^{n-2}\right)$$

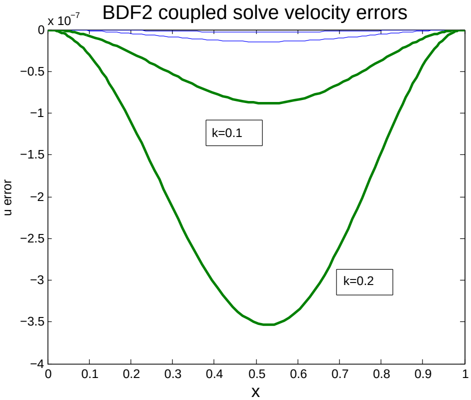
- BDF2 applied to our model can be investigated by solving

$$\begin{aligned} U'' - (1+\beta)U - P' &= f_1 \\ V'' - (1+\beta)V - iP &= f_2 \\ U' + iV &= 0 \end{aligned}$$

where $\beta = (3/2 - 2e^{-ik} + 1/2e^{-2ik})/k = i + O(k^2)$

# Coupled BDF2 Results

- Scaled velocity errors 7.70e-5 ($k = 0.1$), 7.71e-7 ($k = 0.01$).
- Scaled pressure errors 3.26e-5 ($k = 0.1$), 3.27e-7 ($k = 0.01$).
- $O(k^2)$ errors as expected.
- $U = u + k^2 u^{(2)}(x) + O(k^3)$ with $u^{(2)}(x)$ smooth (regular error expansion).



# Basic Projection Method

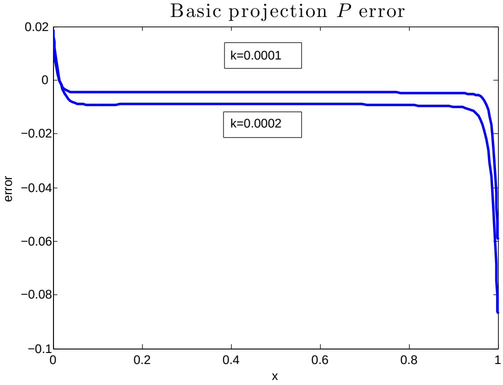
- Backward Euler step without a pressure term giving intermediate velocities that are not divergence free.
- Projection step on the intermediate velocities.
- In our framework:

$$\tilde{U}'' - (1 + 1/k)\tilde{U} + \frac{1}{k}e^{-ik}U = f_1$$
$$\tilde{V}'' - (1 + 1/k)\tilde{V} + \frac{1}{k}e^{-ik}V = f_2$$
$$U = \tilde{U} - kP'$$
$$V = \tilde{V} - ikP$$
$$U' + iV = 0$$

- $V$ is not exactly zero on the boundary.
- $P' = 0$ at boundary points (inconsistent).

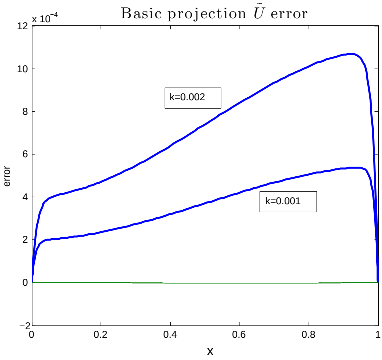
# Basic Projection Method Computational Results

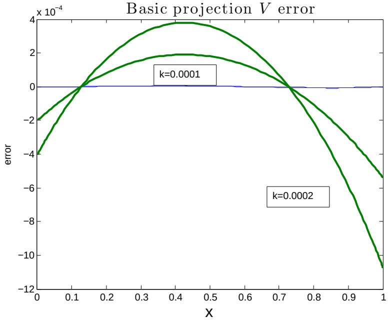
- $\tilde{U}$ errors 5.06e-3 ($k = 1e-4$), 5.11e-4 ($k = 1e-5$).
- $P$ errors 1.09e-2 ($k = 1e-4$), 3.26e-3 ($k = 1e-5$).
- $P = p + kp^{(1)}(x) + \sqrt{k}C_p e^{-x/\sqrt{k}} + \cdots$.



# Basic Projection Method Computations (cont.)

- $\tilde{U} = u + k\tilde{u}^{(1)}(x) + kC_u e^{-x/\sqrt{k}} + \cdots$.
- $\tilde{V}$, $U$ and $V$ have smooth errors at highest order.



# Summary

- Asymptotic error analysis can be used to describe regular errors and numerical artifacts in finite difference methods and other schemes on regular meshes applied to problems with smooth solutions.
- Asymptotic error analysis can be used to help understand the accuracy of different implementations of boundary and interface conditions.