Math 152 – Linear Systems
Test #2, Version B (MWF sections)

Spring, 2010
University Of British Columbia

Name: 

ID Number:

Instructions

• You should have seven pages including this cover.
• There are 2 parts to the test:
  o Part A has 10 short questions worth 1 mark each
  o Part B has 3 long questions worth 5 marks each
• Although all questions in each part are worth the same, some may be much more difficult – do the easy questions first!
• Use this booklet to answer questions.
• Return this exam with your answers.
• Please show your work. Correct intermediate steps may earn credit.
• No calculators are permitted on the test.
• No notes are permitted on the test.
• Maximum score= 25 Marks (attempt all questions)
• Maximum Time= 50 minutes.

GOOD LUCK!

<table>
<thead>
<tr>
<th>Part A</th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>25</td>
</tr>
</tbody>
</table>
Part A - Short Answer Questions, 1 mark each

A1: Consider the matrices and vectors defined below:

\[ A = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix} \]

Which multiplications below are well defined? (circle all that apply)

\[ AA \quad AB \quad AC \quad BC \quad CB \]

A2: Consider the following lines of MATLAB code

```matlab
A = zeros(3,3);
A(1,1) = 5;
A(2,3) = 4;
A(3,3) = -1;
```

What matrix \( A \) will result?

A3: Write the transpose of the matrix below:

\[ A = \begin{bmatrix} 1 & 5 \\ -8 & \sqrt{2} \\ 9 & -2 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & -8 & 9 \\ 5 & \sqrt{2} & -2 \end{bmatrix} \]

A4: Let

\[ A = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} \]

Is \( A \) invertible? (justify briefly).

\[ \text{yes, } \det A \neq 0. \]

A5: Suppose the matrix \( P \) below is the transition matrix for a random walk. Fill in the two missing (underlined) entries below

\[ P = \begin{bmatrix} \underline{4} & \frac{1}{2} \\ \underline{3/4} & \underline{1/2} \end{bmatrix} \]
For questions A6 and A7 below, consider the circuit shown below:

A6: Write a linear equation that describes the sum of the voltage drops round loop 3 (the lower middle loop) in the circuit. Your equation should be in terms of the loop currents and voltages shown in the diagram.

\[ 4i_3 + (i_3 - i_4) + E_2 = 0. \]

A7: Write a linear equation that expresses the current through the 5A current source in terms of the loop currents in the diagram.

\[ i_2 - i_4 = 5 \]

A8: Suppose that \( P \) is a projection, \( R \) a rotation and \( Q \) a reflection (all in 2D). Circle the statement below that is \textit{not} true:

- (a) \( P^2 = P \)
- (b) \( Q^2 = I \)
- (c) \( R^T = R^{-1} \)
- (d) \( Q = Q^T \)
- (e) \( P^T = P^{-1} \)
A9: Let $R$ be the matrix representing counterclockwise rotation by $\pi/2$ (90°) in 2D. Find

$$R \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

A10: What is the matrix representation of reflection through the line $y = x$ in 2D followed by counter-clockwise rotation by $\pi/4$ (45°)?

$$R = \text{Rot } \pi/4.$$ 

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

from formula with angle $\pi/4$ or (easier) reflecting coordinate directions to get the columns of $Q$.

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

formula with angle $\pi/4$.

$$RQ = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

remember this order when $Q$ is done first.
Part B - Long Answer Questions, 5 marks each

B1: Consider the matrix

\[ A = \begin{bmatrix}
1 & 2 & -2 \\
0 & 2 & -1 \\
-2 & 2 & 3
\end{bmatrix} \]

(a) [4 marks] Find \( A^{-1} \). Do the computations using Gaussian Elimination on an augmented matrix for full marks.

(b) [1] Solve

\[ Ax = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \]

for \( x \). Hint: use your result from part (a)

\[ A^{-1} = \begin{bmatrix}
2 & -\frac{5}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \\
\frac{1}{2} & -\frac{3}{2} & \frac{1}{2}
\end{bmatrix} \]

\[ x = (A^{-1})^{-1} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix} \]

(\textit{twice the second column of } A^{-1})
B2: Let $T$ be a linear transformation from three dimensions (3D) to 3D such that

\[
T \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}
\]

(a) [2 marks] Use the information above and the fact that $T$ is linear to determine

\[
T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]

(b) [2] Now determine

\[
T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

(c) [1] What is the matrix representation for $T$?

\[
(a) \quad T \left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) = T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)
\]

\[
= T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) - T \left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)
\]

\[
= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}
\]

(b) \quad T \left( \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) = T \left( \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) - T \left( \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)
\]

\[
= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}
\]

(c) \quad \mathbf{T} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 2 \end{bmatrix}.
B3: A model of the weather in Vancouver is constructed based purely on probabilities. The weather on a given day is said to be either:

1. Sunny
2. Cloudy
3. or Rainy

The weather the next day is predicted based on the weather on the current day. Suppose that the weather the next day will stay the same 50% of the time, and change to the other types 25% each.

(a) [2 marks] Write the transition matrix for Vancouver weather based on the model and numbering of states above.

(b) [2] If it is sunny one day, what is the chance that it will be sunny two days later? Use matrix multiplication to work out this part for full marks.

(c) [1] On average what fraction of the time will it be sunny in Vancouver according to this model? Justify briefly in words.

\[
P = \begin{bmatrix}
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{2}
\end{bmatrix}.
\]

\[
\begin{align*}
\mathbf{x}^{(0)} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\
\mathbf{x}^{(1)} &= \mathbf{P} \mathbf{x}^{(0)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}, \\
\mathbf{x}^{(2)} &= \mathbf{P} \mathbf{x}^{(1)} = \begin{bmatrix} \frac{1}{4} + \frac{1}{16} + \frac{1}{16} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\
\frac{1}{4}
\end{bmatrix}. \\
\end{align*}
\]

so \(\frac{3}{8}\) chance of being sunny after 2 days.

(c) Since all states have the same symmetric behaviors, they will be equally likely and so the weather should be \(\frac{1}{3}\) sunny on average.