Math 152 – Linear Systems
Test #2, Version A (T/Th sections)

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University Of British Columbia

Name: Solutions

ID Number: _____________________________________

Instructions

• You should have seven pages including this cover.
• There are 2 parts to the test:
  o Part A has 10 short questions worth 1 mark each
  o Part B has 3 long questions worth 5 marks each
• Although all questions in each part are worth the same, some may be much more difficult – do the easy questions first!
• Use this booklet to answer questions.
• Return this exam with your answers.
• Please show your work. Correct intermediate steps may earn credit.
• No calculators are permitted on the test.
• No notes are permitted on the test.
• Maximum score= 25 Marks (attempt all questions)
• Maximum Time= 50 minutes.

GOOD LUCK!

<table>
<thead>
<tr>
<th>Part A Total</th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>25</td>
</tr>
</tbody>
</table>
Part A - Short Answer Questions, 1 mark each

A1: Compute the following matrix product
\[
\begin{bmatrix}
1 & -1 & 2 \\
3 & 2 & -1 \\
5 & 1 & 3
\end{bmatrix}
\begin{bmatrix}
0 & 2 \\
-1 & 1 \\
1 & 1
\end{bmatrix}
= \begin{bmatrix}
3 & 3 \\
-3 & 7 \\
2 & 14
\end{bmatrix}
\]

A2: Consider the following lines of MATLAB code
\[
A = zeros(3,3);
for i=1:2
A(i,i) = -2;
A(i,i+1) = 1;
A(i+1,i) = 1;
end
A(3,3) = -1;
\]
What matrix A will result?

A3: Write the transpose of the matrix below:
\[
A = \begin{bmatrix} 1 & 1/2 & -8 \\ 9 & \sqrt{3} & -1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 \\ 9/2 \sqrt{3} \\ -8 \end{bmatrix}
\]

A4: Let \( T \) be a linear transformation from two dimensions (2D) to 3D such that
\[
T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 10/5 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0/3 \end{bmatrix}
\]
Compute
\[
T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = T\begin{bmatrix} 1 \\ 1 \end{bmatrix} - T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 0/3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0/3 \end{bmatrix} = \begin{bmatrix} -3 \\ 10/2 \end{bmatrix}
\]

A5: Consider a random walk with 2 states. There is probability 1/2 that if the system is in state 1 it remains there, probability 1/3 that if the system is in state 2 it remains there. Write the transition matrix \( P \).
\[
P = \begin{bmatrix} 1/2 & 2/3 \\ 1/2 & 1/3 \end{bmatrix}
\]
For questions A6 and A7 below, consider the circuit shown below:

A6: Write a linear equation that describes the sum of the voltage drops round loop 2 (the middle loop) in the circuit. Your equation should be in terms of the loop currents and voltages shown in the diagram.

$$5(i_2 - i_3) - E_1 + 3i_2 = 0.$$  

A7: Write a linear equation that expresses the current through the 1A current source in terms of the loop currents in the diagram.

$$i_2 - i_4 = 1.$$  

A8: Consider now a different circuit with two voltage sources $V_1$ and $V_2$. The fundamental problem for the circuit has been solved and written as

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 3.2 & -0.1 \\ 0.8 & 1.1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where $I_1$ and $I_2$ are the currents through the corresponding voltage sources. If $V_1$ is increased, will $I_2$ increase or decrease? Justify briefly.

$$I_2 = -0.8V_1 + 1.1V_2$$  

so as $V_1$ is increased, $I_2$ will decrease.
A9: If $Q$ is a reflection in 2D, then $Q^2 = I$. Justify this fact briefly. It might help to support your argument with a sketch.

From the picture $Q^2 x = x$ for every $x$ so $Q^2 = I$.

A10: Find two different matrices $B$ such that

$$B^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \text{rotation by } 90^\circ \text{ counter-clockwise.}$$

*Hint:* The RHS matrix above is a rotation.

$$B_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \quad \text{(rotation by } 45^\circ)\]$$

$$B_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \quad \text{can be obtained as } -B_1 \text{ or as rotation by } -135^\circ.$$
Part B - Long Answer Questions, 5 marks each

B1: Consider the matrix

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
1 & 1 & 1 \\
1 & 2 & 1
\end{bmatrix}
\]

(a) [4 marks] Find \( A^{-1} \). Do the computations using Gaussian Elimination on an augmented matrix for full marks.

(b) [1] Solve

\[
Ax = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}
\]

for \( x \). Hint: use your result from part (a)

\[\begin{array}{c|c|c}
& \begin{array}{ccc|c}
1 & 2 & 3 & 1 \\
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 0 \\
\end{array} & \begin{array}{ccc|c}
0 & -1 & -2 & -1 \\
0 & 0 & -2 & -1 \\
0 & -1 & -1 & 1 \\
\end{array} \\
\hline
\end{array}\]

\[\begin{array}{c|c|c}
& \begin{array}{ccc|c}
1 & 2 & 0 & -1/2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1/2 \\
\end{array} & \begin{array}{ccc|c}
1 & 0 & 0 & -1/2 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & -1 \\
\end{array} \\
\hline
\end{array}\]

\[\begin{array}{c|c}
& \begin{array}{cc|c}
1/2 & 10 & 1 \\
-5 & -2 & 1 \\
1/2 & 1 & \end{array} & \begin{array}{c}
10^{1/2} \\
-7 \\
3^{1/2}
\end{array} \\
\end{array}\]
B2: Below, $R$ is the linear transformation of counter-clockwise rotation by $\pi/6$ in two dimensions (2D). $Q$ is reflection through the line $\sqrt{3}y = -x$ in 2D. Let $A$ be the transformation that is $R$ followed by $Q$.

(a) [1 mark] What is the matrix representation of $R$?

(b) [1] What is the matrix representation of $Q$?

(c) [2] What is the matrix representation of $A$?

(d) [1] Compute $A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The following table may be helpful for this question:

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\cos \theta$</th>
<th>$\sin \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\pi/6$ (30°)</td>
<td>$\sqrt{3}/2$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$\pi/4$ (45°)</td>
<td>$\sqrt{1}/2$</td>
<td>$\sqrt{1}/2$</td>
</tr>
<tr>
<td>$\pi/3$ (60°)</td>
<td>$1/2$</td>
<td>$\sqrt{3}/2$</td>
</tr>
<tr>
<td>$\pi/2$ (90°)</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ a) \quad R = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \]
\[ b) \quad Q = \begin{bmatrix} \sqrt{1}/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -\sqrt{1}/2 \end{bmatrix} \]
\[ c) \quad A = Q R \quad \text{(this order for } R \text{ to be done first)} \]
\[ d) \quad A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \]
**B3:** Consider a random walk on four states. The transition matrix $P$ and some of its powers are given below:

$$
P = \begin{bmatrix}
1/2 & 0 & 1/3 & 1/4 \\
1/2 & 1/2 & 1/3 & 1/4 \\
0 & 1/2 & 1/3 & 1/4 \\
0 & 0 & 0 & 1/4
\end{bmatrix}
$$

$$
p^2 \approx \begin{bmatrix}
0.5000 & 0.4444 & 0.3939 & 0.3333 \\
0.4167 & 0.2778 & 0.2708 & 0.2222 \\
0.3333 & 0.3333 & 0.3333 & 0.3333 \\
0 & 0 & 0 & 0
\end{bmatrix}
p^5 \approx \begin{bmatrix}
0.3356 & 0.3314 & 0.3344 & 0.3320 \\
0.3222 & 0.2222 & 0.2222 & 0.2222 \\
0.3356 & 0.3314 & 0.3344 & 0.3320 \\
0 & 0 & 0 & 0.0010
\end{bmatrix}
$$

(a) [2 marks] If the system starts in state 1, what is the probability that it will be in state 2 after two time steps?

(b) [2] If the system is equally likely to start in any of the four states, what state is it most likely to be in after five time steps.

(c) [1] Explain briefly in words why (after looking at $P$) you expect to see the 3 zero entries in the bottom row of $P^2$ and $P^5$.

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(a) read off probability

(b) consider $p^5 \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 
\text{averages of entries in rows of } p^5 
\end{bmatrix}$

Notice that the second row of $p^5$ has the largest values, so State 2 is the most likely after 5 steps.

(c) The three zeros in the last row of $P$ say that you can't enter state 4 from states 1, 2, or 3. If you can't do that in one step, you can't do it in 2 or 5 steps either.