Part A - Short Answer Questions, 1 mark each

For Questions A1-A3 below, let

\[
\begin{align*}
\mathbf{a} &= (1, 2, 2) \\
\mathbf{b} &= (1, -1, -1)
\end{align*}
\]

The answers for questions A1-A4 should be in the form \((x, y, z)\) with \(x, y,\) and \(z\) real numbers.

A1: Find \(3\mathbf{a} + \mathbf{b}\).

\[
(3+1, 6-1, 6-1) = (4, 5, 5)
\]

A2: Find the projection of \(\mathbf{a}\) in the direction of \(\mathbf{b} - \mathbf{a}\).

\[
\frac{\mathbf{a} \cdot (\mathbf{b} - \mathbf{a})}{\|\mathbf{b} - \mathbf{a}\|^2} = -\frac{12}{18} = -\frac{2}{3}
\]

A3: Find a vector \(\mathbf{x} = (x, y, z)\) of length 1 such that \(\mathbf{x} \cdot \mathbf{a} = 0\) and \(\mathbf{x} \cdot \mathbf{b} = 0\).

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 2 & 2 \\
1 & 2 & -1
\end{vmatrix} = (0, 3, -3)
\]

Length \(\sqrt{18} = (0, \frac{3}{\sqrt{18}}, -\frac{3}{\sqrt{18}})\)

\[
\mathbf{x} = (0, \frac{3}{\sqrt{18}}, -\frac{3}{\sqrt{18}})
\]

A4: Evaluate the determinant of the matrix below:

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 0 \\
6 & 0 & 0
\end{vmatrix} = (1)(0) - 2(0) + 3(-30) = -90
\]

A5: The solution set of a linear system with seven variables has the parametric form

\[
\mathbf{x} = \mathbf{a} + t\mathbf{b}
\]

with \(\mathbf{b} \neq 0\). What is the rank of the augmented matrix representing the system?
A6: Consider the following line of MATLAB code:

\[
\text{dot}(a,b) / \text{dot}(b,b)*b
\]

Circle the description below that best describes the action of this code:
(a) Solves a linear system.
(b) Determines a projection.
(c) Determines the normal direction of a plane.
(d) Determines whether a set of vectors is linearly independent.

A7: Find the values of \( a \) and \( b \) such that

\[
L_1 : \quad x = (1, 0, 0) + s(1, 2, 3) \quad \text{and} \quad L_2 : \quad x = (2, 2, a) + t(2, 4, b)
\]

are parametric forms of the same line. Here \( s \) and \( t \) are parameters and \( a \) and \( b \) are values to be determined.

\[
(2, 4, b) \ \text{must be a multiple of} \ (1, 2, 3) \Rightarrow b = 6.
\]

\[
(2, 2, a) : \quad (1, 0, 0) + s(1, 2, 3) \Rightarrow s = 1, \text{ so } a = 3.
\]

A8: Find the volume of the parallelepiped whose corners include \( O = (0, 0, 0) \), \( A = (1, 2, -1) \), \( B = (0, 1, 1) \) and \( C = (0, -1, 2) \) such that \( A, B \) and \( C \) are the corners nearest to \( O \).

\[
\text{det} \begin{pmatrix}
1 & 2 & -1 \\
0 & 1 & 1 \\
0 & -1 & 2
\end{pmatrix} = 3
\]
A9: The vectors $\mathbf{a} = (1, 1, 1)$, $\mathbf{b} = (1, 1, -1)$ and $\mathbf{c} = (2, 2, 4)$ are linearly dependent. Find real numbers $s_1$, $s_2$ and $s_3$ not all zero such that

$$s_1 \mathbf{a} + s_2 \mathbf{b} + s_3 \mathbf{c} = 0.$$ 

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

so $s_3 = t$, $s_2 = t$, $s_1 = -3t$

$$\mathbf{s} = (-3, 1, 1) \quad \text{[or any nonzero multiple]}.$$

A10: Find a parametric form for the set of points in $\mathbb{R}^2$ that are the same distance from points $A = (3, 2)$ and $B = (1, -2)$.

Midpoint $\frac{\mathbf{A} + \mathbf{B}}{2} = (2, 0)$.

$\mathbf{AB} = (1, -2) - (3, 2) = (-2, -4)$

Line direction $\perp$ to $\mathbf{AB}$, take $\mathbf{q} = (-2, 1)$

$$\mathbf{x} = (2, 0) + t (-2, 1)$$
Part B - Long Answer Questions, 5 marks each

B1: Consider the linear system below for the unknowns $x$, $y$, and $z$. It is known that the system has a unique solution.

\[
\begin{align*}
2x + 3y - z &= 16 \\
x + y + 4z &= -8 \\
x + 2y - 6z &= 27
\end{align*}
\]

(a) [1 mark] Write the system in an augmented matrix.

(b) [3] Do row operations (Gaussian elimination) on the augmented matrix to change it to echelon form.

(c) [1] Find the solution to the system.

\[
\begin{align*}
\text{(a)} & \quad \begin{bmatrix} 2 & 3 & -1 & | & 16 \\ 1 & 1 & 4 & | & -8 \\ 1 & 2 & -6 & | & 27 \end{bmatrix} \\
\text{(b)} & \quad \begin{bmatrix} 2 & 3 & -1 & | & 16 \\ 1 & 1 & 4 & | & -8 \\ 1 & 2 & -6 & | & 27 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 4 & | & -8 \\ 2 & 3 & -1 & | & 16 \\ 1 & 2 & -6 & | & 27 \end{bmatrix} \\
& \quad \xrightarrow{R_2 \leftarrow 2R_1} \begin{bmatrix} 1 & 1 & 4 & | & -8 \\ 0 & 1 & -9 & | & 32 \\ 0 & 1 & -10 & | & 35 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_1} \begin{bmatrix} 1 & 1 & 4 & | & -8 \\ 0 & 1 & -9 & | & 32 \\ 0 & 0 & -1 & | & 3 \end{bmatrix} \\
& \quad \xrightarrow{-R_3} \begin{bmatrix} 1 & 1 & 4 & | & -8 \\ 0 & 1 & -9 & | & 32 \\ 0 & 0 & 1 & | & -3 \end{bmatrix} \xrightarrow{z = -3} \begin{bmatrix} 1 & 1 & 0 & | & 32 \\ 0 & 1 & -9 & | & 32 \\ 0 & 0 & 1 & | & -3 \end{bmatrix} \\
& \quad \Rightarrow \begin{align*}
y - 9z &= 32 \\
x + y + 4z &= -8 \\
x &= -8 - (5) - 4(-3) = -1
\end{align*}
\]

\[
\begin{align*}
x &= -1 \\
y &= 5 \\
z &= -3
\end{align*}
\]
B2: The points the \( A = (1, 5, -2), \) \( B = (0, 2, 3), \) and \( C = (1, 1, 1) \) are the vertices of a triangle.

(a) [2 marks] Find a normal vector to the plane that contains the triangle.

(b) [1] What is the area of the triangle?

(c) [1] Write an equation form for the plane \( P \) that contains the triangle.

(d) [1] Is the point \((-2, 5, 7)\) on the plane \( P \) from part (c) above?
Test #1, Version B, Math

B2 (a) To find a normal vector to the plane containing the triangle, we can find two linearly independent vectors in that plane, and take their cross product.

Any two vertices of the triangle will give us a vector in the plane. For example:

AC: \((1, 5, -2) - (1, 1, 1) = (0, 4, -3)\)
BC: \((0, 2, 3) - (1, 1, 1) = (-1, 1, 2)\)

\[
AC \times BC = \begin{vmatrix}
i & j & k \\
0 & 4 & -3 \\
-1 & 1 & 2 \\
\end{vmatrix} = \begin{vmatrix}4 & -3 \\
1 & 2 \\
0 & -3 \\
\end{vmatrix} - \begin{vmatrix}0 & -3 \\
-1 & 2 \\
0 & 4 \\
\end{vmatrix} - \begin{vmatrix}0 & 4 \\
-1 & 1 \\
-1 & 1 \\
\end{vmatrix} = \begin{bmatrix}11 \\
3 \\
4 \\
\end{bmatrix}
\]

Other answers are possible: they will be a scalar multiple of this vector.

(b) Given two vectors \(a, b\) in \(\mathbb{R}^3\), the quantity \(\|a \times b\|\) gives the area of the parallelogram with \(a\) and \(b\) as two of its sides. So, the area of the triangle with those two sides will be \(\frac{1}{2}\|a \times b\|\). That is,

\[
\frac{1}{2} \sqrt{11^2 + 3^2 + 4^2} = \frac{1}{2} \sqrt{146}.
\]

(c) A plane with normal vector \([a, b, c]\) will have equation \(ax + by + cz = d\) for some constant \(d\). So, our plane will have equation \(11x + 3y + 4z = d\) for some constant \(d\). Since we know points on the plane, each of them will have to fulfill the equation. So \(d = 11(1) + 3(1) + 4(1) = 18\). (Equivalently: \(d = 11(1) + 3(5) + 4(-2) = 18\), or \(d = 11(0) + 3(2) + 4(3) = 18\).)

So the equation of the plane is \(11x + 3y + 4z = 18\) (or a scalar multiple of this).

(d) To test whether the point \((-2, 5, 7)\) is on the plane, we plug in \(x = -2\), \(y = 5\), and \(z = 7\) and see whether we get 18.

\(11(-2) + 3(5) + 4(7) = 21 \neq 18\)

so \(\text{no}\) it is not on the plane.