Part A - Short Answer Questions, 1 mark each

A1: Is the matrix
\[ A = \begin{bmatrix} 7 & 5 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix} \]
invertible? Briefly justify your answer.

invertible because \( \det A = 35 \neq 0 \) (or rank \( A = 3 \) or...)

A2: Suppose \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is a linear transformation such that
\[ T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \quad T \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \quad T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \]
What is the matrix representation of \( T \)?
\[ \begin{bmatrix} 1 & 0 & 5 \\ 4 & 2 & 6 \\ 1 & 3 & 7 \end{bmatrix} \]

A3: If \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is a linear transformation and
\[ T \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
what is
\[ T \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) ? \]
\[ T \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) = T \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \]

A4: The linear transformation \( g : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) is given by
\[ g \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 - x_3 \\ -2x_3 \end{bmatrix} \]
Is the vector \( [1 - 2]^T \) in the range of \( g \)? Briefly justify your answer.
\[ g \left( \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ so it's in the range.} \]
A5: A is a $3 \times 3$ matrix typed into MATLAB. It has determinant 4. What is the output of the MATLAB command \texttt{rref}(A)?

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

A6: Three of the following statements about an $n \times n$ matrix $A$ are equivalent. Circle the statement below that is not equivalent to the other three.

(a) $A$ is invertible.
(b) $	ext{det}(A) \neq 0$.
(c) The rows of $A$ are linearly dependent.
(d) The rank of $A$ is $n$.

A7: Consider a random walk with 2 states. The probability of staying at either state is 1/2. If the walker is in state 1 initially, what is the probability that he will be in state 2 after 3 time steps?

$$
P = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

$$
x_1 = Px_0 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad x_2 = Px_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

probability $\frac{1}{2}$. 
A8: For the circuit in Figure 1, write the linear equation that corresponds to Kirchhoff's voltage law applied to loop 2 (that is, the loop that corresponds to the loop current $i_2$ in the figure). Write your equation in terms of the loop currents and voltage drops in the figure.

$$\dot{i}_2R_2 + E + \dot{i}_2R_3 - V = 0$$

A9: For the circuit in Figure 1, write the linear equation that corresponds to the match of loop currents to the current source $I$.

$$\dot{i}_3 - \dot{i}_2 = I$$

A10: For the circuit in Figure 1, write the current $J$ through the voltage source in terms of the loop currents.

$$J = \dot{i}_2 - \dot{i}_4.$$
Part B - Long Answer Questions, 5 marks each

B1: Compute the inverse of

\[ A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{bmatrix} \]

\[ r_2 \leftrightarrow r_3. \]

\[ \text{new } r_2 + \text{new } r_3. \]

\[ \begin{bmatrix} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{bmatrix} \]

\[ r_1 - 2 \text{new } r_3. \]

\[ r_3. \]

\[ A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} \]

(easy to check if \( A^{-1} \) is correct).
B2: A tourist visits a tropical island. She spends each day in one of the four cities on the island. If she is in the first city one day, she will move to the second or third with equal probability but will not stay in the first city or go to the fourth. If she is in the second city, she is equally likely to be in any city the next day. If she is in the third city, she will move to the first or fourth city with equal probability but will not stay in the third city or go to the second. If she is in the fourth city she stays there half of the time or goes to the second city the other half of the time. Her movement between the cities is considered as a random walk with transition probability matrix $P$. Some MATLAB computations with $P$ determine that

$$P^4 \approx \begin{bmatrix}
0.2109 & 0.1680 & 0.1250 & 0.1641 \\
0.2634 & 0.3320 & 0.3750 & 0.3359 \\
0.1484 & 0.1680 & 0.1875 & 0.1641 \\
0.3316 & 0.3320 & 0.3125 & 0.3359 
\end{bmatrix} \quad \text{and} \quad \lim_{n\to\infty} P^n \approx \begin{bmatrix}
0.1667 & 0.1667 & 0.1667 & 0.1667 \\
0.3333 & 0.3333 & 0.3333 & 0.3333 \\
0.1667 & 0.1667 & 0.1667 & 0.1667 \\
0.3333 & 0.3333 & 0.3333 & 0.3333 
\end{bmatrix}$$

(a) [2 marks] Write the matrix $P$.

(b) [1] If the tourist starts in the first city, what is the chance she will be in the third city after 4 days?

(c) [1] If the tourist starts in the first city, what is the chance she will be in the third city after a long time on the island?

(d) [1] Explain briefly why the answer to the last question does not depend on the location where the tourist began her stay.

\[ \begin{align*}
\text{a)} \quad & \quad P = \begin{bmatrix}
0 & \frac{1}{4} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{2} \\
0 & \frac{1}{4} & \frac{1}{2} & 0
\end{bmatrix} \\
\text{b)} \quad & \quad x_4 = P^4 \cdot x_0 = P^4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1484 \\ 0.3320 \\ 0.3320 \\ 0.1680 \end{bmatrix} \text{ chance to be in 3rd city.} \\
\text{c)} \quad & \quad \text{As above, chance is } 0.1667 = (P^\infty)_{31}. \\
\text{d)} \quad & \quad \text{Because the columns of } P^\infty \text{ are identical, in particular all entries in the 3rd row are the same.}
\end{align*} \]
B3: Consider the 2D projection $P$, rotation $R$ and reflection $F$ given in terms of their matrix representations below:

\[
P = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}, \quad R = \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix}, \quad F = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}
\]

(a) [1] Evaluate

\[F\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\]

(b) [1] Find all vectors $x$ such that $P(x) = 0$.

(c) [1] What is the matrix representation of $T(x) = R(F(x))$ (the composition of $R$ and $F$)?

(d) [2] What is the equation of the line onto which $P$ projects? Write the line in the form $y = ax$ with $a$ determined.

\[
\begin{align*}
\text{(a)} & \quad \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -7/5 \end{bmatrix} \\
\text{(b)} & \quad \begin{bmatrix} 1/5 & 4/5 \\ 2/5 & 4/5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{so} \quad x = s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ for any scalar } s. \\
\text{(c)} & \quad T = RF = \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix} \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 7/25 \end{bmatrix} \\
\text{(d)} & \quad P\left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix} \text{ must be on the line,} \\
\text{so} \quad y = 2x \text{ is the equation.}
\]