Math 152 – Linear Systems
Test #1, Version A

Spring, 2009
University Of British Columbia

Name: _______________________________________________________

ID Number: __________________________________________________

Instructions

• You should have six pages including this cover.
• There are 2 parts to the test:
  o Part A has 10 short questions worth 1 mark each
  o Part B has 3 long questions worth 5 marks each
• Use this booklet to answer questions.
• Return this exam with your answers.
• Please show your work. Correct intermediate steps may earn credit.
• No calculators are permitted on the test.
• No notes are permitted on the test.
• Maximum score= 25 Marks (attempt all questions)
• Maximum Time= 50 minutes.

GOOD LUCK!

<table>
<thead>
<tr>
<th>Part A Total</th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>25</td>
</tr>
</tbody>
</table>


Part A - Short Answer Questions, 1 mark each

A1: Are $(1, 3, 1)$ and $(0, -1, 2)$ orthogonal? Briefly justify your answer.

\[
(1, 3, 1) \cdot (0, -1, 2) = -3 + 2 = -1 \neq 0 \text{ not orthogonal}
\]

A2: For what value of the scalar $s$ are the vectors $(1, 2, 3)$ and $(-2, -4, s)$ collinear?

\[s = -6.\]

A3: Consider two points $(a_1, a_2)$ and $(b_1, b_2)$ in the plane. Find an expression for the midpoint of the line segment joining these two points.

\[
\left( \frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2) \right).
\]

A4: Are the vectors $(1, 2)$ and $(3, 1)$ linearly independent? Briefly justify your answer.

\[
\text{different directions, not a multiple of the other, } \det \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = -5 \neq 0 \text{ (any of these reasons), they are l.i.}\n\]

A5: Consider the following line of MATLAB code:

\[
A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix};
\]

Pick the case below that describes the object $A$ assigned by the code:

(a) a matrix with 2 rows and 3 columns.
(b) a matrix with 3 rows and 2 columns.
(c) a row vector with 6 components.
(d) the statement results in an error message.

A6: It is possible for a linear system of 2 equations in 3 unknowns to have (circle the correct answer below):

(a) only a single solution.
(b) a single solution or no solutions.
(c) no solutions or an infinite number of solutions.
(d) a single solution or an infinite number of solutions.
A7: Give an example of a $2 \times 2$ matrix with no zero entries that has a zero determinant.

\[
\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \text{ many other choices.}
\]

A8: Consider the following lines of MATLAB code that modify a previously defined $3 \times 9$ matrix $A$

```matlab
for i=1:3
    A(i,:) = A(i,:)*i;
end
```

Circle the case below that correctly describes the action of this code:

(a) Exchanges the first and third rows of $A$.
(b) Multiplies each row of $A$ by its row number.
(c) Multiplies the first three columns of $A$ by their column numbers.
(d) Multiplies the third row of $A$ by 3, leaving the other rows unchanged.

A9: Consider a linear system with two equations for two unknowns $x_1$ and $x_2$. It is known that the system has a single vector solution with components $x_1 = 2$ and $x_2 = 3$. What is the reduced row echelon form of the augmented matrix that represents the system?

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 3
\end{bmatrix}
\]

A10: For problem #9 above, circle the correct geometric interpretation of the system:

(a) two lines in the plane that intersect at a single point.
(b) two parallel but different lines in the plane.
(c) two identical lines in the plane.
(d) the intersection of two planes in three dimensional space.
Part B - Long Answer Questions, 5 marks each

B1: Let \( A = (1, 2, 1) \), \( B = (2, -2, 3) \) and \( C = (0, -3, 1) \) be three points in three dimensional space. Consider the triangle \( T \) whose vertices are \( A \), \( B \) and \( C \).

(a) [1 mark] What is the length of the side \( AB \)?

(b) [4] Find an equation of the plane \( P \) containing \( T \) in the form

\[ ax + by + cz = d \]

with \( a, b, c, d \) determined.

\[ \begin{align*}
\text{a)} \quad \overrightarrow{AB} &= (2, -2, 3) - (1, 2, 1) = (1, -4, 2) \\
\text{length is} &\quad \sqrt{1^2 + (-4)^2 + 2^2} = \sqrt{21} \\
\text{b)} \quad \overrightarrow{AC} &= (0, -3, 1) - (1, 2, 1) = (-1, -5, 0) \\
\text{plane normal} \quad \mathbf{n} &= \overrightarrow{b} \times \overrightarrow{c} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & -4 & 2 \\ -1 & -5 & 0 \end{pmatrix} \\
&= (10, -2, -9). \\
\mathbf{n} \cdot (\mathbf{x} - \mathbf{A}) &= 0 \\
\Rightarrow &\quad 10x_1 - 2x_2 - 9x_3 = 10 \cdot 1 - 2 \cdot 2 - 9 \cdot 1 = -3.
\end{align*} \]
B2: Find all solutions (if any) to the system of equations

\[
\begin{align*}
x_1 + x_2 + x_3 &= 2 \\
2x_1 + x_3 &= 1 \\
3x_1 + 2x_2 + x_3 &= 4
\end{align*}
\]

\[
\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 3 & 2 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & -1 & -3 \\ 0 & -1 & -2 & -2 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 \frac{1}{2} & 3\frac{1}{2} \\ 0 & 0 -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 2 \div (-2) \\ (3) + \text{new}(2) \end{bmatrix}
\]

\[
\begin{align*}
x_3 &= 1/3, \\
x_2 + \frac{1}{2} \left( \frac{1}{3} \right) &= 3/2, \Rightarrow x_2 = \frac{3}{2} - \frac{1}{6} = 4/3. \\
x_1 + \left( \frac{4}{3} \right) + \left( \frac{1}{3} \right) &= 2, \Rightarrow x_1 = 1/3.
\end{align*}
\]

\[
\begin{align*}
x &= \left( \frac{1}{3}, \frac{4}{3}, \frac{1}{3} \right).
\end{align*}
\]
B3: Consider the linear system of equations with parameters $a$ and $b$:

\[
\begin{align*}
2x_1 + 3x_2 &= 10 \\
x_1 + ax_2 &= b
\end{align*}
\]

(a) [3 marks] For what value(s) of $a$ and $b$ does the system have a unique solution?

(b) [1] For what value(s) of $a$ and $b$ does the system have an infinite number of solutions?

(c) [1] For what value(s) of $a$ and $b$ does the system have no solutions?

\[
\begin{bmatrix} 2 & 3 & 10 \\ 1 & a & b \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{3}{2} & 5 \\ 0 & a - \frac{3}{2} & b - 5 \end{bmatrix}
\]

(2) - new (1).

a) when $a \neq \frac{3}{2}$ there is a unique solution for any $b$.

b) when $a = \frac{3}{2}$, $b = 5$ there is an infinite number of solutions.

c) when $a = \frac{3}{2}$, $b \neq 5$, there are no solutions.