0.1 Notes on Hysteresis

Consider the problem
\[ y' = y - \frac{1}{3} y^3 + \lambda. \] (1)

This has equilibriums when
\[ \lambda = -y + \frac{1}{3} y^3, \] (2)

which are shown in figure 1. The reason why we obtain hysteresis is that as \( \lambda \) increases from some large negative value, it stays on the negative stable branch until \( \lambda = \lambda_c = 2/3 \), whereas as \( \lambda \) decreases from some large positive value, it stays on the positive branch until \( \lambda = -\lambda_c \). If we let
\[ \lambda = \lambda_c \cdot 1.2 \cdot \sin (0.1t), \] (3)

then the solution will continuously go through this hysteresis loop. This is shown in figure 2.

0.2 Delay in Bifurcation

Note that in figure 2, there is a small delay between the disappearance of the, for example, positive stable stationary solution and the time when the solution moves to the negative stable stationary solution. The object of this section is to find the approximate length of that delay.

Let \( \lambda = \lambda_c + \epsilon t \). Expanding everything nearby, \( y = y_c + \epsilon^0 y_1 \) and \( t = \epsilon^0 \tau \). As a result,
\[ y' = f(y, \lambda) \] (4)

\[ = f|_{y_c, \lambda_c} + f_y|_{y_c, \lambda_c} (y - y_c) + f_{yy}|_{y_c, \lambda_c} \frac{(y - y_c)^2}{2} + f_{\lambda}|_{y_c, \lambda_c} (\lambda - \lambda_c). \] (5)
Figure 2: The solution to the equation 1 where $\lambda$ varies in time as in equation 3. The initial condition that was used is $y(0) = -0.1$. 
Now by definition, when the stable stationary solution first ‘disappears’, \( f = 0 \) since we’re still at the stationary solution. Locally at the point where the stable stationary solution first ‘disappears’, the curve determined by \( f(y, \lambda) = 0 \) is vertical in the \( \lambda \)-y plane, and so \( f_y = 0 \). As a result we obtain

\[
y' = f_{yy} \frac{(y - y_c)^2}{2} + f_{\lambda y} \frac{(\lambda - \lambda_c)}{2}
\]

(6)

\[
e^{p-q} y_{1\tau} = e^{2p} \frac{f_{yy}}{2} y_1^2 + e^{1+q} f_{\lambda \tau}.
\]

(7)

In order for these to all be of the same order, we must have

\[
p - q = 2p = 1 + q
\]

(8)

and as a result, we must have \( p = 1/3, q = -1/3 \). We then obtain that

\[
y_{1\tau} = \frac{f_{yy}}{2} y_1^2 + f_{\lambda \tau},
\]

(9)

which is an equation of order 1. We must now solve this equation. If we let \( y_1 = \beta v \) and \( \tau = \delta s \), then we obtain

\[
\frac{\beta}{\delta} v' = \frac{f_{yy}}{2} v^2 + f_{\lambda} \delta s
\]

(10)

\[
v' = \frac{f_{yy}}{2} v^2 + \frac{f_{\lambda} \delta^2}{\beta} s.
\]

(11)

Now we choose to specify that

\[
\frac{f_{yy} \beta \delta}{2} = -1, \quad \frac{f_{\lambda} \delta^2}{\beta} = 1,
\]

(12)

which means that

\[
\delta = -\left( \frac{2}{f_{yy} f_{\lambda}} \right)^{1/3}, \quad \beta = \left( \frac{4 f_{\lambda}}{f_{yy}^2} \right)^{1/3},
\]

(13)

and so

\[
v' = -v^2 + s.
\]

(14)

If we then let \( v(s) = \varphi'(s)/\varphi(s) \), we obtain that

\[
\varphi''(s) = s \varphi(s),
\]

(15)

and so

\[
\varphi(s) = a_0 \text{Ai}(s) + a_1 \text{Bi}(s),
\]

(16)

\[
v(s) = \frac{a_0 \text{Ai}'(s) + a_1 \text{Bi}'(s)}{a_0 \text{Ai}(s) + a_1 \text{Bi}(s)}.
\]

(17)
Now as \( t \to -\infty \), by definition, \( s \to \infty \). Now as \( s \to \infty \), \( \text{Ai}(s) \to 0 \) and \( \text{Bi}(s) \to \infty \), and so

\[
v(s) \xrightarrow{s \to \infty} \begin{cases} 
\frac{\text{Bi}(s)}{\text{Bi}(s)} & \text{if } a_1 \neq 0 \\
\frac{\text{Ai}(s)}{\text{Ai}(s)} & \text{if } a_1 = 0 
\end{cases}
\]

\[
\xrightarrow{s \to \infty} \begin{cases} 
\sqrt{s} & \text{if } a_1 \neq 0 \\
-\sqrt{s} & \text{if } a_1 = 0 
\end{cases}
\] (18)

Now in our case, our initial conditions (that we started from \( \lambda = -\infty, y = -\infty \)) require that as \( s \to \infty \), \( v(s) \) must be negative. As a result, we conclude that \( a_1 = 0 \). We then obtain that

\[
y = y_c + e^p y_1
\]

\[
y = y_c + e^{1/3} \beta v(s)
\]

\[
y = y_c + e^{1/3} v(s)
\]

\[
y = y_c + e^{1/3} \left( \frac{4f\lambda}{f_{yy}} \right)^{1/3} \frac{\text{Ai}'(s)}{\text{Ai}(s)}
\]

(20)

(21)

(22)

(23)

where

\[
s = \frac{\tau}{\delta}
\]

\[
= -\tau \left( \frac{f_yf_{yy}}{2} \right)^{1/3}
\]

\[
= -te^{-q} \left( \frac{f_yf_{yy}}{2} \right)^{1/3}
\]

\[
= -te^{1/3} \left( \frac{f_yf_{yy}}{2} \right)^{1/3}
\]

(24)

(25)

(26)

(27)

Now we expect this to be invalid when \( y_1 \) becomes large. As a result, we predict that the solution will approach the opposite stationary branch when \( \text{Ai}(s) = 0 \) for the first time, at which point \( y_1 \) blows up. This occurs when \( s \sim -2.33811 \).

**No Time Dependence**

Set \( \lambda = \lambda_c + \epsilon \) (with no time dependence). Then expand \( y \) and \( t \) using \( y = y_c + e^p y_1 \) and \( t = e^q \tau \). Then, performing analysis as before,

\[
f(y, \lambda) = y - \frac{1}{3} y^3 + \lambda
\]

\[
y' = \frac{f_{yy}}{2} (e^p y_1)^2 + f_\lambda \epsilon
\]

\[
e^{p-q} y_{1\tau} = e^{2p} \frac{f_{yy}y_1^2}{2} + f_\lambda \epsilon
\]

so that \( p - q = 2p = 1 \) means that \( p = 1/2, q = -1/2 \) and

\[
y_{1\tau} = \frac{f_{yy}}{2} y_1^2 + f_\lambda.
\]

(28)

(29)

(30)

(31)
Solving this, we find that

\[ y = y_c + \sqrt{\frac{2f_\lambda}{f_{yy}}} \tan \left( \tau \sqrt{\frac{f_{yy} f_\lambda}{2}} \right), \]  

which we expect to be valid until

\[ \tau \sim \frac{\pi}{\sqrt{2f_{yy} f_\lambda}} \]  
\[ t \sim \frac{\pi}{\sqrt{2f_{yy} f_\lambda}} \frac{1}{\sqrt{\epsilon}} \]  

As a result, even when we speed up this process by letting \( \lambda - \lambda_c \) increase linearly with time, this natural delay causes the solution to take a while before it moves over to the opposite stationary branch.

### 0.2.1 Simple Example

If we consider the example from the introduction (§0.1), that

\[ y' = y - \frac{1}{3} y^3 + \lambda, \]  

then \( f_\lambda = 1 \) and \( f_{yy} = -2y_c = 2 \). From equations 23 and 27 we have that

\[ y = -1 + \epsilon^{1/3} \frac{Ai'(s)}{Ai(s)}, \]  
\[ s = -t \epsilon^{1/3}. \]  

We expect a transition to occur when \( Ai(s) = 0 \), which is when \( s \sim -2.33811 \), or

\[ t \sim \epsilon^{-1/3} \cdot 2.33811 \]  

This approximate solution is shown in figure 4 for \( \epsilon = 0.01 \), in comparison with the numerical solution also shown in figure 3.

### 0.2.2 Insect Infestation

#### Introduction

A model for the spruce budworm infestation is

\[ \frac{dN}{dt} = RN \left( 1 - \frac{N}{k} \right) - P(N) \]  

where \( N \) is the size of the population, \( R \) is the maximum rate of growth of the population, \( k \) is the carrying capacity of the environment and \( P(N) \) is the rate of death (predation rate) of the population. The predation rate is given by

\[ P(N) = \frac{BN^2}{A^2 + N^2}, \]  

5
Figure 3: The numerical solution to equation 1 with $\epsilon = 0.01$ and $\lambda = \lambda_c + ct$. The initial condition is $y = -2$ at $t = -150$.

Figure 4: A zoomed-in version of the numerical solution to equation 1 with $\epsilon = 0.01$ and $\lambda = \lambda_c + ct$, as in figure 3. This is shown in comparison with the approximate solution given by equation 36.
which goes through the origin and has a maximum as $N \to \infty$ of $B$. If we let

$$x = \frac{1}{A} N, \quad \tau = \frac{B}{A} t, \quad r = \frac{A}{B} R, \quad \kappa = \frac{1}{A} k$$

then our problem reduces to the non-dimensional problem

$$\frac{dx}{d\tau} = rx \left(1 - \frac{x}{\kappa}\right) - \frac{x^2}{1 + x^2}. \quad (42)$$

The equilibrium points for this are shown for $r = 0.55$ in figure 5. For $r$ fixed we vary $\kappa$ according to

$$\kappa = \left(\frac{\kappa_1 + \kappa_2}{2}\right) + 2 \times \left(\frac{\kappa_2 - \kappa_1}{2}\right) \sin(\epsilon t), \quad (43)$$

where $\kappa_1$ and $\kappa_2$ are the lower and upper bounds on the region where there is an unstable equilibrium point. We then obtain the solutions shown in figures 6 and 7. Observe that even though in figure 7 there are regions of time in which the region of low population is no longer stable, due to the time delay required to reach the region of high population, the solution is forced to remain in the low-population state.

**Number of Solutions**

The equilibriums of equation 42 occur when

$$0 = rx \left(1 - \frac{x}{\kappa}\right) - \frac{x^2}{1 + x^2} \quad (44)$$

$$r \left(1 - \frac{x}{\kappa}\right) = \frac{x}{1 + x^2}. \quad (45)$$
Figure 6: The solution to equation 42 for \( r = 0.55 \) and where \( \kappa \) varies in time as in equation 43. Here, \( \epsilon = 0.01 \) and the initial condition that was used is \( x(0) = 10 \).
Figure 7: The solution to equation 42 for \( r = 0.55 \) and where \( \kappa \) varies in time as in equation 43. Here, \( \epsilon = 0.05 \) and the initial condition that was used is \( x(0) = 10 \).
Figure 8: The curve and line from equation 46, with \( r \) fixed and different values of \( \kappa \). This shows the possibility for 1, 2 or 3 equilibrium points.

By letting

\[
y = \frac{x}{1 + x^2} = r \left(1 - \frac{x}{\kappa}\right),
\]

we can plot these and see where they intersect. See figure 8. For the value of \( r \) shown, there are two values of \( \kappa \), labelled \( \kappa_1 \) and \( \kappa_2 \) in the figure, such that there are two equilibriums. As a result, a bifurcation occurs at \( \kappa_1 \) and \( \kappa_2 \) for this value of \( r \). For \( \kappa_1 < \kappa < \kappa_2 \), there are three equilibriums, and for either \( \kappa < \kappa_1 \) or \( \kappa > \kappa_2 \), there is only one equilibrium. In order to determine the region in the \( r\kappa \) plane in which there are three equilibriums, we then search for the values of \( r \) and \( \kappa \) such that the line is tangent to the curve. This will form the border of the region in question.

If the line is tangent to the curve at some point \((x_0, y_0)\), then \( r \) and \( \kappa \) must satisfy

\[
\frac{x_0}{1 + x_0^2} = r \left(1 - \frac{x_0}{\kappa}\right),
\]

\[
\frac{1 - x_0^2}{(1 + x_0^2)^2} = \frac{r}{\kappa}.
\]

By solving for these, we obtain the curve parametrically in terms of \( x_0 \):

\[
r = \frac{2x_0^3}{(x_0^2 + 1)^2}, \quad \kappa = \frac{2x_0^3}{x_0^2 - 1}.
\]

By graphing this, we obtain figure 9.

Observe that, as in figure 10, that the smallest value of \( x_0 \) for which a line can be (almost) tangent to the curve is 1, where for a line to be tangent to the curve, it would need to have \( r = 0.5, \kappa = \infty \). Since the curve here is
concave down, as $x_0$ increases past 1, the value of $r$ will rise until $x_0$ reaches the point of inflection, $x_c$, after which $r$ will descend again. As a result, the maximum value of $r$ occurs when the line is tangent to the curve at the point of inflection. We can calculate that this occurs when

$$x_c = \sqrt{3}, \quad r_c = \frac{3\sqrt{3}}{8}, \quad \kappa_c = 3\sqrt{3}. \quad (50)$$

**Approximation to Delay**

Here,

$$f_\kappa = \frac{r x^2}{\kappa^2} = 0.0051 \quad (51)$$

$$f_{xx} = \frac{-2r}{\kappa} - \frac{2 - 6x^2}{(1 + x^2)^3} = 0.39 \quad (52)$$

If we let $\kappa = \kappa_c + \epsilon t$ for $r$ fixed, the above analysis holds, with $y$ replaced with $x$ and $\lambda$ replaced with $\kappa$. We then expect a transition to occur when $\text{Ai}(s) = 0$, which is when $s \sim -2.33811$, or

$$t \sim \epsilon^{-1/3} \cdot 23.4768 \quad (53)$$

The approximate solution obtained from equation 23 is shown in figure 12 for $\epsilon = 0.01$, in comparison with the numerical solution also shown in figure 11.
Figure 10: The curve and line from equation 46, with the critical values of $r$ and $\kappa$ given in equation 50.

Figure 11: The numerical solution to equation 42 with $\epsilon = 0.01$, $r = 0.55$ and $\kappa = \kappa_c + \epsilon t$. The initial condition is $y = 1$ at $t = -600$. 
Figure 12: A zoomed-in version of the numerical solution to equation 42 with $\epsilon = 0.01$, $r = 0.55$ and $\lambda = \lambda_c + ct$, as in figure 11. This is shown in comparison with the approximate solution given by equation 23.