A MEMS Capacitor

"Micro-Electrical-Mechanical System"

We will derive a nonlinear PDE for the deflection of the top membrane due to an imposed voltage bias across the thin gap of width \( l \).

Potential between membrane and bottom plate

\[
\begin{align*}
\Delta' \Psi_1 &= 0 \quad &1 x' \leq L/2, &1 y' \leq W/2 \\
\Psi_1 &= 0 \quad &\text{on } z' = 0.
\end{align*}
\]

Inside the membrane

\[
\nabla' \cdot \left[ \varepsilon_2 \nabla' \Psi_2 \right] = 0 \quad &1 x' \leq L/2, &1 y' \leq W/2 \\
\Psi_2 &= \Psi' \text{ on } z' = u'(x', y') + d
\]

Now the continuity conditions are

\[
\Psi_1 = \Psi_2 \quad \text{at } z' = u'(x', y') - d
\]

\[
\varepsilon_2 \nabla' \Psi_2 \cdot \hat{n} = \varepsilon_0 \nabla' \Psi_1 \cdot \hat{n} \quad \text{at } z' = u'(x', y') - d
\]

Here \( \varepsilon_2 = \varepsilon_2 (x', y') \) is the dielectric permittivity of the membrane, and \( \varepsilon_0 \) is the permittivity of air.

Here \( \hat{n} \) is the unit normal to the surface \( z' = u'(x', y') + d = 0 \).

And so

\[
\hat{n} = \left( -u'_x, -u'_y, 1 \right) / \sqrt{1 + (u'_x)^2 + (u'_y)^2}
\]
Now we assume that $W, L \gg \ell$ where $\ell$ is the gap width between the membrane and the bottom plate.

Now let $x = x'/L$, $y = y'/L$, $z = z'/\ell$, $u = u'/\ell$.

The equations become:

In gap

$$
\begin{cases}
\varepsilon_{1zz} + \frac{\sigma^2}{L^2} (\varepsilon_{1xx} + \varepsilon_{1yy}) = 0 & |x| < \frac{L}{2}, \ |y| < \frac{W}{2L} \\
\psi_1 = 0 & \text{on } z = 0
\end{cases}
$$

Let $\varepsilon_2(x, y) = \varepsilon_2(x/L, y/L)$. Then,

$$
\begin{cases}
\varepsilon_{2zz} + \frac{\sigma^2}{L^2} \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \cdot \left[ \varepsilon_2 \left( \frac{\partial \psi_1}{\partial x}, \frac{\partial \psi_1}{\partial y} \right) \right] = 0 & \text{in } |z - u| \leq \frac{d}{4}, \ |x| \leq \frac{L}{2}, \ |y| \leq \frac{W}{2L} \\
\varepsilon_2 \left( \frac{1}{L} \frac{\partial}{\partial x}, \frac{1}{L} \frac{\partial}{\partial y}, \frac{1}{\ell} \frac{\partial}{\partial z} \right) \left( -\frac{1}{L} u, -\frac{1}{L} u, 1 \right) \psi_2 \\
= \varepsilon_0 \left( \frac{1}{L} \frac{\partial}{\partial x}, \frac{1}{L} \frac{\partial}{\partial y}, \frac{1}{\ell} \frac{\partial}{\partial z} \right) \left( -\frac{1}{L} u, -\frac{1}{L} u, 1 \right) \psi_1 \\
\psi_1 = \psi_1 & \text{on } z = u - \frac{d}{4} \\
\psi_2 = V & \text{on } z = u + \frac{d}{4}
\end{cases}
$$

Now in the limit $1/L \ll 1$, (thin gap), we obtain

Membrane

$$
\begin{cases}
\varepsilon_{2zz} = 0 & u - \frac{d}{4} \leq z \leq u + \frac{d}{4} \\
\psi_2 = V & z = u + \frac{d}{4}
\end{cases}
$$

Gap

$$
\begin{cases}
\varepsilon_{1zz} = 0 & 0 \leq z \leq u - \frac{d}{4} \\
\psi_1 = 0 & z = 0
\end{cases}
$$
The continuity conditions are
\[
\varepsilon_2 \Psi_{2z} = \varepsilon_0 \Psi_{1z} \quad \text{on} \quad z = u - d/l
\]
\[
\Psi_1 = \Psi_2 \quad \text{on} \quad z = u - d/l
\]

Define \( z_L = u - d/l \), \( z_u = u + d/l \).

We solve to get
\[
\Psi = \begin{cases} 
A_1 z & \text{on} \quad 0 \leq z \leq z_L \\
A_2 (z - z_u) + V & \text{on} \quad z_L < z < z_u 
\end{cases}
\]

Now continuity gives:
\[
A_1 z_L = A_2 (z_L - z_u) + V
\]
\[
\varepsilon_2 A_2 = \varepsilon_0 A_1
\]

This yields that
\[
A_1 = \frac{\varepsilon_1/\varepsilon_0}{(\varepsilon_1/\varepsilon_0) z_L + 2d/l} \quad \frac{V}{[((\varepsilon_1/\varepsilon_0) z_L + 2d/l]}
\]

This yields the thin gap result:
\[
\Psi = \begin{cases} 
\frac{(\varepsilon_1/\varepsilon_0) V}{(\varepsilon_1/\varepsilon_0) z_L + 2d/l} z & 0 \leq z \leq z_L \\
\frac{V}{[((\varepsilon_1/\varepsilon_0) z_L + 2d/l]} (z - z_u) + V & z_L < z < z_u 
\end{cases}
\]

Now a deformable elastic membrane satisfies
\[
\nabla^2 u'(x', y') = \frac{\varepsilon_2}{2} \left| \nabla' \Psi_2 \right|^2 \quad \text{on} \quad z' = u'(x', y') - d.
\]

where \( T \) is the tension in the membrane.

Next, we will assume even further that \( d/l << 1 \). Then on the surface \( z = U \) we have:
\[
\Psi_2 \sim V \left( \frac{\varepsilon_0}{\varepsilon_2} \right) \left( \frac{z - U}{U} \right) + V
\]

Now writing \( x = x'/L, y = y'/L, u = u'/l \) we obtain that
\[ T \left[ \frac{q}{L^2} \right] (U_{xx} + U_{yy}) = \frac{\varepsilon_2}{2} \left( \Psi_{zz} \right)^2 + \ldots \]

This yields with \[ \Psi_{zz} = V \frac{\varepsilon_0}{\varepsilon_2} \frac{1}{U} \] that

\[ \begin{cases} U_{xx} + U_{yy} = \frac{\lambda F(x,y)}{U^2} & \text{in } \Omega \\ U = 1 & \text{on } \partial\Omega \end{cases} \]

with \[ \lambda = \frac{\varepsilon_0 L^2 V^2}{2 L^3 T} \]

\[ F(x,y) = \frac{\varepsilon_0}{\varepsilon_2(x,y)} \]

\[ \lambda \text{ bifurcation parameter.} \]

This is the equation for the deflection \( U \) of the membrane.

Notice that \[ \lambda \propto V^2 \] which is the square of the applied voltage. Notice that when \( U = 0 \) we have a singularity as the membrane touches the ground plate.

Questions

(i) What is \( U_{\min} \) vs \( \lambda \)? Typically we get a picture of the form:

\[ U_{\min} \]

\[ \lambda^* = \text{fold point value} \]

\[ V_{\text{pull-in}} = \left( \frac{\lambda^*}{C} \right)^{1/2} \]

(ii) We would like to estimate \( \lambda^* \) and in particular try to choose, or optimize \( \varepsilon_2 \), so that \( \lambda^* \) is large. This will allow steady-state solutions for a large interval in \( V \).
MEMS PDE PUT LOWER PLATE AT $U = -1 \Rightarrow \Omega \rightarrow \Omega$ $U : 0$

THEN WE WRITE (4) $\Delta U = \frac{\lambda f(x)}{(1+U)^2}$ IN $\Omega$

$U = 0$ ON $\partial \Omega$

WITH $f(x) = \frac{\varepsilon_0}{\varepsilon_1(x)}$ IN $\Omega$, WITH $\varepsilon_0 > 0$

AND $\lambda = b \sqrt{V^2}$ THE BIFURCATION PARAMETER. WE REQUIRE $U > -1$ IN $\Omega$.

**PROPOSITION** $\exists A^* > 0$ SUCH THAT NO SOLUTION TO (4) EXISTS IF $\lambda > \lambda^*$.

I.E. A MAXIMUM VOLTAGE $V_{pull} = \sqrt{\frac{\lambda^*}{b}}$ SUCH NO SOLUTION EXISTS IF $V > V_{pull}$.

\[ \lambda^* \]

\[ \lambda^* \]

\[ \lambda^* \]

**PROOF** SUPPOSE THAT $U$ EXISTS. THEN BY GREEN'S IDENTITY WITH $\Delta \phi_i = -\sigma_i \phi_i$, WE OBTAIN

\[ \int_{\Omega} (\phi_i, \Delta U - U \Delta \phi_i) \, dx = \int_{\Omega} \phi_i f \, dx + \int_{\Omega} \phi_i U \sigma_i \, dx = 0 \]

WE CONCLUDE THAT $I = \int_{\Omega} (\frac{\lambda f}{(1+U)^2} + U \sigma_i) \phi_i \, dx = 0$

THUS IF A SOLUTION EXISTS, WE MUST FIND THAT $I = 0$.

NOW $I \geq \int_{\Omega} \frac{\lambda c}{(1+U)^2} + U \sigma_i \phi_i \, dx$

SO IF $\frac{\lambda c}{(1+U)^2} + U \sigma_i > 0 \quad \forall -1 < U$ THEN $I > 0$ AND WE HAVE A CONTRADICTION $\rightarrow$ NO SOLUTION.

LET $f = \frac{1}{(1+U)^2}$ $g = -\sigma_i U$.

IF $f > g$ ON $U > -1 \rightarrow$ NO SOLUTION.
WE CONCLUDE THAT $F > g$ ON $U > 1$

**iff** $A > A^*$ WHERE $A^*$ SATISFY THE TANGENCY CONDITION

$$\frac{1}{(1+U)^2} = -\frac{\sigma_1}{A_c}$$

$$\frac{-2}{(1+U)^3} = -\frac{\sigma_1}{A_c}$$

**divide** $\frac{3}{2} = U \rightarrow 3U = -1, U = -\frac{1}{3}$

Thus

$$A^* = \frac{\sigma_1}{2c} \left(1+U^3\right) = \frac{4\sigma_1}{27}$$

**Example** Consider the unit disk with $F = 1$. Then $\phi_i = J_0(\sqrt{\sigma_1}, r)$ with $J_0(\sqrt{\sigma_1}) = 0$ or $\sigma_1 = (2.4048...)^2 \rightarrow A^* = \frac{4}{27} (2.4048)^2 \approx 8.57$

**Example** in slab domain $-\frac{1}{2} < x < \frac{1}{2}$ then $\phi_i = \sin(\sqrt{\sigma_1}(x + \frac{1}{2}))$

So $\sin(\sqrt{\sigma_1}) = 0 \rightarrow \sqrt{\sigma_1} = \pi$ or $\sigma_1 = \pi^2$. We have for $c = 1$ that

$$A^* = \frac{4}{27} \left(\pi^2\right) \approx 1.46$$

**Remark** Method of proof needs $C > 0$, i.e. $F$ Bounded Away From Zero.

**1-D Problem**

$$u^{\prime\prime} = \frac{\lambda}{(1+U)^2}, \quad 1 < x < \frac{1}{2}$$

$u(x) = u(-x)$, $u$ convex, $u(x) \leq 0$ on $1 | x | < 1/2$

$u \left( \pm \frac{1}{2} \right) = 0$

We put $u(0) = u_0$ with $-1 < u_0 < 0$ and calculate:
We multiply by \( u' \) and integrate

\[
\frac{u'^2}{2} = \frac{A}{1+u} + C = \frac{A}{1+u_0} - \frac{A}{1+u} = \frac{A(1-u_0)}{(1+u_0)(1+u)}
\]

We get

\[
u' = \sqrt{2 \frac{A}{1+u_0}} \left[ \frac{u - u_0}{(1+u_0)(1+u)} \right]^{1/2}
\]
on \( 0 < x < \frac{1}{2} \).

Integrating yields:

\[
\int \frac{(1+u)^{1/2}}{(u-u_0)^{1/2}} \, du = \int \frac{\sqrt{2A}}{(1+u_0)^{1/2}} \, dx
\]

so

\[
\int_{u_0}^{u} \frac{(1+u)^{1/2}}{(u-u_0)^{1/2}} \, d\lambda = \frac{\sqrt{2A}}{\sqrt{1+u_0}} \, x
\]

Now \( u \left( \frac{1}{2} \right) = 0 \) yields

\[
\sqrt{\frac{A}{2(1+u_0)}} = \int_{0}^{u_0} \sqrt{\frac{1+\lambda}{\lambda-u_0}} \, d\lambda = 2 \int_{0}^{\sqrt{-u_0}} \sqrt{1+\lambda + s^2} \, ds.
\]

Now let \( \lambda - u_0 = S^2 \). 

\[
\sqrt{\lambda} = \frac{2S}{\sqrt{\lambda-u_0}} \, ds = 2 \sqrt{\lambda-u_0} \, ds
\]

IN THIS WAY WE OBTAIN

\[
\sqrt{\lambda} = 2 \sqrt{2(1+u_0)} \int_{0}^{\sqrt{-u_0}} g(u_0, s) \, ds \quad g(u_0, s) = \sqrt{1+u_0 + s^2}
\]

NOTICE THAT \( \sqrt{\lambda} = 0 \) WHEN \( u_0 = 0 \) AND WHEN \( u_0 = -1 \). MOREOVER, \( \sqrt{\lambda} > 0 \) ON \( -1 < u_0 < 0 \). THUS \( \Delta \) A MAX. VALUE FOR \( \lambda \), LABELED BY \( \lambda_{\text{MAX}} \), AS \( u_0 \) RANGES ON \( -1 < u_0 < 0 \).

NOW WE CALCULATE \( \lambda \) \( u_0 \to 0 \) THAT

\[
\sqrt{\lambda} \sim 2 \sqrt{2(1+u_0)} \left( \sqrt{u_0} \right) \quad \text{SO} \quad \lambda \sim \beta (u_0)^{+ -}
\]
Now \( u_0 \to -1 \) we have \[
\sqrt{\frac{2}{\lambda + 2}} \sqrt{1 + u_0} \int_0^1 g(-1, s) \, ds = \frac{1}{2} \sqrt{2} \sqrt{1 + u_0}.
\]
Thus \( \lambda \sim 2 (1 + u_0) \) at \( u_0 \to -1 \).

\[\|u\| = -u_0\]

\[\lambda_c = 1.4, \quad \lambda^*\]

By a numerical evaluation \( d\lambda / du_0 = 0 \) at \( \lambda = \lambda_c \approx 1.4 \).

Recall the existence threshold bound \( \lambda^* \). We found that \( \lambda^* \approx 1.46 \).

**Radial Case**

Suppose \( f = 1 \).

We have \( u'' + \frac{1}{\lambda} u' = \frac{A}{\Gamma (1 + \lambda)^2}, \quad u(1) = 0, \quad u'(0) = 0. \)

\( J_0(z_0) = 0, \quad z_0 = 2.4048 \)

We would like to plot \( -u(0) \) versus \( \lambda \). No solution if \( \lambda > \frac{A}{27} (z_0)^2 \approx 9.857 \).

We now convert to an ODE:

Set \( u = -1 + \alpha w(p) \quad p = \gamma \Gamma \).

We can take \( w(0) = 1 \) and \( w'(0) = 0 \).

Substitute: \( \alpha \gamma \frac{1}{2} w'' + \frac{\alpha \gamma^2}{p} w' = \frac{A}{\alpha^2 \gamma^2} \)

Now choose \( \lambda = \alpha^3 \gamma^2 \), then we have the ODE IVP:

\[
\begin{align*}
&\begin{cases} 
  w'' + \frac{1}{p} w' = \frac{1}{w^2} \\
  w(0) = 1, \quad w'(0) = 0
\end{cases}
\end{align*}
\]
\[ u(1) = 0 \rightarrow -1 + \alpha w(\gamma) = 0 \rightarrow \alpha = \frac{1}{w(\gamma)}. \]

This yields with \( \alpha = \nu(0) + 1 \) that

\[ \lambda = \frac{\gamma^2}{[w(\gamma)]^3} \]

\[ \nu(0) + 1 : \quad \lambda = \frac{1}{w(\gamma)} \]

or

\[ ||u|| = \nu(0) : \quad \lambda = 1 - \frac{1}{w(\gamma)} \] (2)

\[ \lambda = \frac{\gamma^2}{[w(\gamma)]^3}. \]

This yields a parameterization of the radially symmetric steady states in the \( \nu(0) - \lambda \) plane.

N\underline{umerically this is now trivial and we find that} \( \lambda_c = 0.789. \)

\textbf{Remark Numerical:} We must have \( w \sim 1 + A \rho^2 \) at \( \rho \to 0 \)

so that from PDF \( 2A + 2A = 1 \rightarrow A = \frac{1}{4}. \)

Let \( \varepsilon = 0.001 \) \( w(\varepsilon) = 1 + \varepsilon^2/4, \quad w'(\varepsilon) = \varepsilon/2. \)

Now we must explain the bifurcation diagram. The key feature observed numerically is the infinite number of saddle-node bifurcation points.
We will consider long-time behavior for
\[ w'' + \frac{1}{\rho} w' = \frac{1}{w^2}, \quad w(0) = 1, \quad w'(0) = 0. \]

We put \( \eta = \log \rho \) so that \( w(\rho) \to \tilde{w}(\eta) \).

Then
\[ \tilde{w}_\rho = \frac{\tilde{w}}{\rho / \rho}, \quad \tilde{w}_{\rho \rho} = \frac{\tilde{w}}{\rho / \rho} - \frac{\tilde{w}_\rho}{\rho^2} \to \tilde{w}_\eta = \frac{1}{\tilde{w}^2}, \quad \tilde{w}_\eta = \frac{e^{2\lambda}}{\tilde{w}^2}. \]

Now let \( \tilde{w} = e^{B\lambda} \eta \to \tilde{w}'' = B^2 e^{B \lambda} \eta + 2B e^{B \lambda} \eta' + e^{B \lambda} \eta'' \) so
\[ \eta'' + 2B \eta' + B^2 \eta = \frac{e^{2B \lambda} \eta}{e^{2B \lambda} \eta^2} = \frac{e^{2B \lambda}}{\eta^2}. \]

Thus \( B = 2/3 \) and we get
\[ \begin{aligned}
\eta'' + \frac{4}{3} \eta' + \frac{4}{9} \eta &= \frac{1}{\eta^2}, \\
The \text{system is a 2-dof dynamical system: } v_1: v_1', v_2: v_2 \to v_2' = F(v),
\end{aligned} \]

v_2' = v_1',
\[ v_1' = -\frac{4}{3} v_1 + \frac{4}{9} v_2 + \frac{1}{v_2^2}. \]

Now the equilibrium is \( \frac{4}{9} \frac{v_e}{v_e^2} = \frac{1}{v_e} \to v_e = \frac{9}{4}, \text{ or } v_e = \left(\frac{9}{4}\right)^{1/3}. \)

Now linearize: \( v = v_e + \phi \) with \( \phi \ll 1 \)

Then
\[ \phi'' + \frac{4}{3} \phi' + \frac{4}{9} \frac{1}{v_e^2} \phi = -\frac{2}{v_e^3} \phi \to \phi'' + \frac{4}{3} \phi' + \frac{4}{9} \phi + \frac{8}{9} \phi = 0. \]

So
\[ \phi'' + \frac{4}{3} \phi' + \frac{12}{9} \phi = 0. \]

Now put \( \phi = e^{\lambda t} \to \lambda^2 + \frac{4}{3} \lambda + \frac{12}{9} = 0, \) so \( \lambda = -\frac{2}{3} + \frac{2\sqrt{2}}{3} \).
THIS YIELDS 
\[ \phi = A e^{-2/3 \lambda} \cos \left( \frac{2\pi}{3} \lambda + \phi_0 \right) \] 
\[ \phi_0 \] phase shift.

Now recall \( \lambda = \log \rho \), then for \( \lambda \gg 1 \) we have

\[ \lambda = \left( \frac{9}{4} \right)^{1/3} A e^{-2\lambda/3} \cos \left( \frac{2\pi}{3} \lambda + \phi_0 \right) \] 
\[ A < c \]

with \( \lambda = \log \rho \), \( w = \rho^{1/3} \) \( v \), so \( \rho^{1/3} e^{-2\lambda/3} = 1 \) we get

\[ w \sim \left( \frac{9}{4} \right)^{1/3} \rho^{1/3} + A \cos \left( \frac{2\pi}{3} \log \rho + \phi_0 \right) \] 
\[ \text{as} \ \rho \to \infty \].

RECALL
\[ u(0) = 1 - \frac{1}{w(y)} \], \( \lambda(y) = \frac{y^2}{(w(y))^3} \)

THUS
\[ u(0) \sim 1 - \frac{1}{\left( \frac{9}{4} \right)^{1/3} y^{1/3} + \ldots} \], \( \lambda \sim \frac{y^2}{\left( \frac{9}{4} \right)^{1/3} y^{1/3} \left[ 1 + \frac{A}{\lambda^{1/3}} \cos \left( \ldots \right) \right]^3} \)

THUS YIELDS FOR \( y \to \infty \) THAT
\[ u(0) \to 1 \]

AND
\[ \lambda \sim \frac{A}{\lambda^{1/3}} \left[ 1 - \frac{3A}{\lambda^{1/3}} \left( \frac{4}{9} \right)^{1/3} \cos \left( \frac{2\pi}{3} \log \lambda + \phi_0 \right) \right] \]

\[ u(0) \]

DIELECTRIC PERMITTIVITY

WE LET \( \frac{\varepsilon_0}{\varepsilon_2} = \Gamma \) \( \alpha \) so that \( \varepsilon_2 = \varepsilon_0 \Gamma^{-\alpha} \) \( (\text{HIGH PERMITTIVITY NEAR } \Gamma = 0) \)

THEN IN A DISK WE HAVE

\[ u_{\Gamma} \Gamma + \frac{1}{\Gamma} u_{\Gamma} = \frac{\lambda}{(1 + u)^2} \] 
\[ u(1) = 0, \ u'(0) = 0 \] WITH \( \alpha > 0 \).

NOW IF WE LET \( u(\Gamma) = -1 + \alpha w(\rho) \) WITH \( \rho = \gamma \Gamma \) WE GET
\[ \gamma^2 \alpha \left( W^{1/2} + \frac{1}{\rho} w' \right) = \frac{\Lambda}{\gamma^{2/3} \alpha^2 w^2} \]
This yields,

\[ W'' + \frac{1}{\rho} W' = \frac{A \rho^\mu}{\delta^{2+\mu} \alpha^3} \frac{1}{W^2}. \]

Choose \( W(0) = 1 \) WLOG. Then take

\[ \lambda = \frac{\delta^{2+\mu}}{\alpha^3}. \]

And \( W(1) = 0 \) so \(-1 + \lambda W(\lambda) = 0\) or \(\alpha = \frac{1}{W(\lambda)}.\)

Then we get

\[ \lambda = \frac{\delta^{2+\mu}}{[W(\lambda)]^3}. \]

\[-\lambda(0) = 1 - \frac{1}{W(\lambda)}.\]

We want to plot \(-\lambda(0)\) versus \(\lambda\) for different \(\mu.\)

\[ \begin{array}{c}
\lambda \\
-\lambda(0)
\end{array} \]

Now to find the long-time \(\gamma \to \infty\) behavior we let

\[ \lambda = \log \rho \quad \text{and} \quad W = \rho^\beta \sqrt{v}, \quad \text{this yields}\]

\[ \rho^{B-2} v'' + (2B-1) \rho^{B-2} v' + B(B-1) \rho^{B-2} v + \rho^{B-2} v' + B \rho^{B-2} v = \frac{A-2B}{v^2}. \]

Choose \(B-2 = A-2B\) \(\rightarrow\) \(B = (\rho+2)/3.\)

\[ \frac{v''}{3} + 2 \frac{(A+2)}{3} v' + \frac{(A+2)^2}{9} v = \frac{1}{v^2}. \]

Equilibrium \(v_e = \left( \frac{9}{(A+2)} \right)^{1/3}.\)
Now if we substitute \( V = V_e + e^\Delta \Lambda \) we get

\[
\frac{\Delta^2}{3} + \frac{2}{3} (\beta + 2) \Lambda + \left( \frac{N + 2}{q} + \frac{2}{V_e} \right) = 0
\]

or

\[
\frac{\Delta^2}{3} + \frac{2}{3} (\beta + 2) \Lambda + \left( \frac{N + 2}{q} + 2 \frac{(\mu + 2)^2}{q} \right) = 0
\]

Thus we have

\[
\frac{\Delta^2}{3} + \frac{2}{3} (\mu + 2) \Lambda + \frac{(\mu + 2)^2}{3} = 0.
\]

We get either negative or complex conjugate root with negative real part.

\[
\Delta = -\frac{N + 2}{3} + \frac{1}{2} \sqrt{\frac{4}{9} (\mu + 2)^2 - \frac{A(\mu + 2)^2}{3}} = \frac{-\mu + 2}{3} + \frac{i\sqrt{2}}{3} (\mu + 2)
\]

Thus

\[
V \sim \left( \frac{q}{(2 + \mu)^2} \right)^{1/3} + A e^{-\frac{(\mu + 2)\lambda}{3}} \cos \left( \frac{\sqrt{2}}{3} (\mu + 2) \Lambda + \phi_0 \right)
\]

Now in terms of \( W \) we have

\[
W \sim \rho^{(\mu + 2)/3} \left( \frac{q}{(2 + \mu)^2} \right)^{1/3} + A \cos \left( \frac{\sqrt{2}}{3} (\mu + 2) \log \rho + \phi_0 \right)
\]

Now

\[
\lambda = \frac{\gamma^2 + \mu}{(W(\gamma))^3}, \quad W(10) = 1 - \frac{1}{W(\gamma)}
\]

Thus

\[
\Lambda \sim \Lambda_{\infty} = \frac{(\mu + 2)^2}{9} A, \quad \gamma \to \infty.
\]
Let \( \phi_0(x) > 0 \) and \( \mu_0 \) be first eigenpair of Laplacian, i.e.
\[
\Delta \phi_0 + \mu_0 \phi_0 = 0 \quad \text{in } \Omega \\
\phi_0 = 0 \quad \text{on } \partial \Omega
\]
which we normalize as \( \int_{\Omega} \phi_0 \, dx = 1 \).

We first state and prove Jensen's inequality:

**Lemma (Jensen's Inequality)**: Let \( \phi_0(x) > 0 \) in \( \Omega \) with \( \int_{\Omega} \phi_0 \, dx = 1 \). Assume that \( g(u) \) is convex \( \forall u \). Define \( F = \int_{\Omega} \phi_0 \, dx \). Then
\[
g\left( \int_{\Omega} \phi_0 \, dx \right) \leq \int_{\Omega} g(F) \, dx.
\]

**Proof**: Let \( L = a + bF \) be the straight line tangent to \( g(F) \) at the point \( F = \bar{F} \).

Then by convexity, \( g(F) \geq a + bF \), \( \forall F \).

Now multiply by \( \phi_0(x) > 0 \) in \( \Omega \) and integrate
\[
\int_{\Omega} g(F) \phi_0 \, dx \geq \int_{\Omega} \left( a + b \bar{F} \right) \phi_0 \, dx \geq a \int_{\Omega} \phi_0 \, dx + b \int_{\Omega} \phi_0 \, dx.
\]

Thus
\[
\int_{\Omega} g(F) \phi_0 \, dx \geq a + b \bar{F} = g(\bar{F}).
\]

Then we have
\[
g\left( \int_{\Omega} \phi_0 \, dx \right) \leq \int_{\Omega} g(F) \phi_0 \, dx.
\]

\[\rightarrow\] This means
\[
g\left( \int_{\Omega} \phi_0 \, dx \right) \leq \int_{\Omega} g(F) \phi_0 \, dx.
\]

We will use this as
\[
\int_{\Omega} g(F) \phi_0 \, dx \geq g\left( \int_{\Omega} \phi_0 \, dx \right).
\]
NOW CONSIDER THE PDE
\[ \Delta u = \frac{A}{(1 + u)^2}, \quad x \in \Omega \]
\[ u = 0 \text{ on } \partial \Omega \]
AND ASSUME FOR SIMPLICITY THAT \( u = 0 \) AT \( t = 0 \).

PROPOSITION 
Suppose that \( A > \frac{4A_0}{27} \). Then \( u \) reaches \(-1\) at a finite time (i.e. the top plate collapses onto bottom plate in finite time).

PROOF 
Define \( E(t) = \int_{\Omega} u \phi_0 \, dx \). Notice that \( E(0) = 0 \).

THEN, MULTIPLY PDE BY \( \phi_0 \) AND INTEGRATE:
\[ \phi_0 u_t = \phi_0 \Delta u - \frac{A}{(1 + u)^2} \phi_0 \]
Define \( g(u) = \frac{1}{(1 + u)^2} \). So \( g''(u) = \frac{6}{(1 + u)^4} > 0 \).

WE GET
\[ \frac{d}{dt} \int_{\Omega} u \phi_0 \, dx = \int_{\Omega} \phi_0 \Delta u \, dx - \lambda \int_{\Omega} g(u) \phi_0 \, dx \]
\[ = \int_{\Omega} u \phi_0 \, dx - \lambda \int_{\Omega} g(u) \phi_0 \, dx \]
\[ = \int_{\Omega} u \phi_0 \, dx - \lambda \int_{\Omega} g(u) \phi_0 \, dx. \]

THEOREFORE, \( E(t) = \int_{\Omega} u \phi_0 \, dx \). THEN, BY JENSEN,
\[
\begin{bmatrix}
\frac{dE}{dt} + \mu_0 E = -\lambda \int_{\Omega} g(u) \phi_0 \, dx \\
E(0) = 0
\end{bmatrix}
\]

NOW \( E(t) \geq (\inf u) \int_{\Omega} \phi_0 \, dx = (\inf u). \)
Now we have the problem
\[
\frac{dE}{dt} + \mu_0 E \leq -\frac{A}{(1+E)^2}
\]
with \(E(0) = 0\) and \(E(t) \geq \inf U\).

Now define \(F(t)\) to be the solution to the ODE
\[
\frac{dF}{dt} + \mu_0 F = -\frac{A}{(1+F)^2}
\]
with \(F(0) = 0\).

By standard comparison principle \(E(t) \leq F(t)\) for all \(t\).

Proof: Define \(v = F - E\). Then
\[
\frac{dv}{dt} + \mu_0 v \geq -\frac{A}{(1+F)^2} + \frac{A}{(1+E)^2} = \frac{\Lambda \left[ (1+F)^{-2} - (1+E)^{-2} \right]}{(1+E)^2 (1+F)^2} = \frac{\Lambda \int_0^t 2 (f-E) \left(f-E\right) \left[f(E)-E\right]}{(1+E)^2 (1+F)^2}.
\]

Or
\[
\frac{dv}{dt} + \mu_0 v \geq \lambda v \cdot h(t) \quad \text{with} \quad h(t) = \frac{\lambda \left[ 2 + (f+E) \right]}{(1+E)^2 (1+F)^2} \geq 0.
\]

Now
\[
(e^{\mu_0 t} v)' \geq \lambda e^{\mu_0 t} h(t) \quad \text{with} \quad v(0) = 0.
\]

Thus
\[
e^{\mu_0 t} v \geq \lambda \int_0^t v(\tau) h(\tau) e^{\mu_0 \tau} d\tau > 0.
\]

This shows that with \(E(t) \geq \inf U\) and \(E(t) \leq F(t)\) that
\[
F(t) > \inf U.
\]

Now if we can show that \(F(t)\) reaches \(-1\) in finite time

then so will \(\inf U\).
Hence if $F$ has a "touchdown" time $T_F$, then the touchdown time for $U$ is at $T_x$ where $T_x < T_F$.

Now we find range of $\lambda$ for which $F(T_F)$.

We separate variables in (1) to get

$$
\frac{dF}{\mu_0 F + \frac{\lambda}{(1 + F)^2}} = -dt.
$$

Integrating with $F(0) = 0$ we get setting $F(T_F) = -1$ that

$$
T_F = \int_{-1}^{0} \frac{ds}{(\mu_0 s + \frac{\lambda}{(1 + s)^2})}.
$$

If this integral is finite, then the touchdown time $T_x$ for $U$ satisfies

$$
T_x < \int_{-1}^{0} \frac{ds}{(\mu_0 s + \frac{\lambda}{(1 + s)^2})}.
$$

This integral is finite iff the denominator never vanishes.

Now

$$
\frac{1}{(1 + s)^2} > -\frac{\mu_0 s}{\lambda}
$$

This is the tangency argument before, so

$$
\lambda > \lambda_x = \frac{4}{27} \mu_0 \quad \text{and} \quad T_x < \int_{-1}^{0} \frac{ds}{(-1 \mu_0 s + \frac{\lambda}{(1 + s)^2})}.
$$
In conclusion, when $\lambda > I_*$ where we can guarantee no steady-state solutions, it follows that the solution to

$$u_t = \Delta u - \frac{1}{(1+u)^2} \text{ in } \Omega$$

$$u = 0 \text{ on } \partial \Omega; \quad u = 0 \text{ at } t = 0$$

is such that $\inf_{\Omega} u(t) \to 0$, singularity in finite time (quenching).
LINEAR STABILITY ANALYSIS

We begin by writing $u_\xi = \Delta u - \frac{A}{f(u)}$ in $\Omega$. Let $\tilde{u}$ be steady-state solution so that $\Delta \tilde{u} = \frac{1}{f(\tilde{u})}, \quad x \in \Omega$

with $\tilde{u} = 0$ on $\partial \Omega$.

The linearization yields $\Delta \Phi = \frac{A}{f(\tilde{u})} \Phi + \sigma \Phi$ in $\Omega$

$s = 0$ on $\partial \Omega$.

We have that $s_j, s_j$ for $j = 1, 2, \ldots$ are real with $s_j = 1, 2, \ldots$. For linear stability we must show that $s_j < 0$ so that $\sigma > 0$.

Now consider for simplicity a ball domain in $\mathbb{R}^N$ so that $\Delta \tilde{u} = \tilde{u}_{gg} + \frac{(N-1)}{r} \tilde{u}_r$ with either $N = 1$ or $N = 2$.

Label $\tilde{u}(0) = -s$ so that $0 < s < 1$. We have shown that

\[
\begin{cases}
\text{N:1} & \text{N:2} \\
N:1 & N:2
\end{cases}
\]

The minimal branch is the one for which $dA/ds > 0$ on $0 < s < S_0$.

Notice that at the simple fold point we have $A'(S_0) = 0$ with $A'(s) < 0$ for $s > S_0$ (small) and $A'(s) > 0$ for $s < S_0$ (small).

The problem for $\tilde{u}(r, s)$ is

\[
\left\{ \begin{array}{l}
\tilde{u}_{rr} + \frac{(N-1)}{r} \tilde{u}_r = \frac{A}{(1+\tilde{u})^2} \quad \text{in } 0 < r < 1 \\
\tilde{u}(0, s) = -s, \quad \tilde{u}_r(0, s) = 0, \quad \tilde{u}(1, s) = 0.
\end{array} \right.
\]
This yields the solution \( \bar{u}(\bar{\tau}, s) \) and \( \Lambda(s) \).

We now label \( \Lambda_N \bar{u} = \bar{u}_{\bar{\tau}} + \frac{(N-1)}{2} \bar{u}_{s} = \Gamma^{N-1} \left[ \Gamma^N \bar{u}_{\Gamma} \right]_{\bar{\tau}} = \frac{\Lambda}{(1+\bar{\tau})^2} \)

We differentiate w.r.t. \( s \) to obtain

\[ \Lambda_N \bar{u}_s - \Lambda_f \bar{u} \bar{u}_s = \Lambda_s \bar{u}_s \]

with \( f(\bar{u}) = \frac{1}{(1+\bar{\tau})^3} \), \( f'(\bar{u}) = -\frac{2}{(1+\bar{\tau})^3} \).

\( \bar{u}_s(0; s) = -1, \quad \bar{u}_s(0; s) = 0, \quad \bar{u}_s(1; s) = 0. \)

Now we consider the eigenvalue problem

\[
\begin{align*}
\Lambda_N \bar{u} - \Lambda f(\bar{u}) \bar{u} &= \sigma \bar{u} \\
\bar{u}_\Gamma &= 0 \text{ at } \Gamma = 0; \quad \bar{u} = 0 \text{ at } \Gamma = 1.
\end{align*}
\]

We then use Lagrange to obtain

\[
\int_{\Omega} \left( \Lambda_N \bar{u}_s - \Lambda f(\bar{u}) \bar{u}_s \right) - \bar{u}_s \left( \Lambda_N \bar{u} - \Lambda f(\bar{u}) \bar{u} \right) \, dx = \int_{\Omega} \left( \bar{u}_s \bar{u}_s - \bar{u}_s \bar{u} \bar{u}_s \right) \, dx = 0.
\]

This yields that

\[
\int_{\Omega} \bar{u}_s \bar{u}_s \, dx = \sigma \int_{\Omega} \bar{u}_s \bar{u} \, dx.
\]

We conclude that

\[
\sigma \int_{\Omega} \bar{u}_s \bar{u} \, dx = \Lambda_s \int_{\Omega} f(\bar{u}) \, dx,
\]

where \( f(\bar{u}) = \frac{1}{(1+\bar{\tau})^2} \). Therefore, at a fold point \( \Lambda_s = 0 \) we have

that the linearized problem \((\dagger)\) has a zero eigenvalue. We observe

the following:

(i) Suppose that \( \bar{u}_s < 0 \) on \( 0 < \bar{\tau} < 1 \). Then applying \((\dagger)\) with \( \bar{u} = \bar{u}_s > 0 \)

we conclude that \( \sigma, < 0 \) if \( \Lambda_s > 0 \) \( \rightarrow \) stable

\( \sigma, > 0 \) if \( \Lambda_s < 0 \) \( \rightarrow \) unstable.

(ii) From the PDE can we infer anything about \( \bar{u}_s \)?

(iii) If \( \bar{u}_s < 0 \) on \( 0 < \bar{\tau} < 1 \) then \( \bar{u}_s = -\bar{u}_s > 0 \) on \( 0 < \bar{\tau} < 1 \)

and the fold point is due to exchange of stability.
IN 1-D IF \( U_5 < 0 \) ON \( 0 < \xi < 1 \) WE HAVE

\[
\begin{array}{c}
\sigma \\
\int_0^1 u \phi \, dx = \Lambda \int_0^1 \phi \frac{f(\nu)}{\nu} \, dx
\end{array}
\]

THE PROBLEM FOR \( u, u_{xx} = \frac{\Lambda}{(1 + \nu)^2} \)

\( u(0) = -S, \ u_x(0) = 0, \ u\left(\frac{1}{2}\right) = 0. \)

INTEGRATING ONCE

\[ \frac{u_x}{2} = \frac{-\Lambda}{1 + \nu} + \frac{\Lambda}{1-S} = \frac{(\nu + S) \Lambda}{(1 + \nu)(1-S)} \]

NOW

\[ u_x = \sqrt{2\Lambda} \left( \frac{\nu + S}{(1 + \nu)(1-S)} \right)^{1/2} \]

\[ \left( 1 + \nu \right)^{1/2} \int u \, dx = \frac{\sqrt{2\Lambda}}{\sqrt{1-S}} \int \frac{1}{(\nu + S)^{1/2}} \, d\nu \]

NOW INTEGRATING

\[ \int \frac{1}{(\nu + S)^{1/2}} \, d\nu = \frac{2\nu}{1-S} \]

LET \( \Lambda + S = t^2 \) THEN

\[ (\Lambda + S)^{1/2} = t \]

AND \( d\nu = 2t \, dt \).

THUS

\[ \frac{d\nu}{(\Lambda + S)^{1/2}} = 2 \]

AND

\[ 2 \int_0^\infty \frac{\sqrt{\nu + S}}{\sqrt{1-S + t^2}} \, dt = \frac{\sqrt{2\Lambda}}{\sqrt{1-S}} \int_0^\infty \frac{\sqrt{\nu + S}}{\sqrt{1-S + t^2}} \, dt. \]
Now let \( t = (1 - s)^{1/4} \lambda \), \( dt = (1 - s)^{1/4} d\lambda \).

Then
\[
x \sqrt{\lambda} = \sqrt{2(1-s)^{3/2}} \int_0^1 (1-s)^{1/2} \sqrt{1+\lambda^2} \, d\lambda.
\]

We conclude that
\[
x \sqrt{\lambda} = \sqrt{2} (1-s)^{3/4} \int_0^1 \sqrt{1+\lambda^2} \, d\lambda.
\]

With
\[
U(u,s) = \left( \frac{u+s}{1-s} \right)^{1/4} = \left( -1 + \frac{u+1}{u-1} \right)^{1/4}
\]

Now when \( x = \frac{1}{2}, \ u = 0 \) so that
\[
\sqrt{\lambda} = 2 \sqrt{2} (1-s)^{3/4} \int_0^1 \sqrt{1+\lambda^2} \, d\lambda.
\]

Define \( \lambda \) as a function of \( s \).

Now we would like to establish the sign of \( U_s \) on \( 0 < x < \frac{1}{2} \).

We calculate
\[
\frac{1}{2} x \lambda^{-1/4} A_s = - \frac{3}{2} \sqrt{2} (1-s)^{1/2} \int_0^1 \sqrt{1+\lambda^2} \, d\lambda + \sqrt{2} (1-s)^{3/4} \sqrt{1+u^2} U_s.
\]

Thus
\[
x \lambda^{1/4} A_s + \sqrt{2} (1-s)^{1/2} \int_0^1 \sqrt{1+\lambda^2} \, d\lambda = \sqrt{2} (1-s)^{3/4} \sqrt{1+u^2} U_s.
\]

This yields
\[
U_s = \frac{1}{\sqrt{2} (1-s)^{3/4} \sqrt{1+u^2}} \left[ x \frac{A_s}{\sqrt{\lambda}} + \sqrt{2} (1-s)^{1/4} \int_0^1 \sqrt{1+\lambda^2} \, d\lambda \right].
\]

Now
\[
2 U_s = U_s (1-s)^{-1} + (u+1)(1-s)^{-2}
\]

so
\[
U_s (1-s)^{-1} + (u+1)(1-s)^{-2} = \frac{1}{2 \sqrt{2} U (1-s)^{3/4} \sqrt{1+u^2}} \left[ x \frac{A_s}{\sqrt{\lambda}} + \sqrt{2} (1-s)^{1/4} \int_0^1 \sqrt{1+\lambda^2} \, d\lambda \right].
\]

We would like to claim that \( U_s < 0 \) on this branch.
NOW FROM (4) WE DEVELOP A FORMULA TO HOLD AT ANY FOLD POINT WHERE \( \lambda_j = 0 \) BUT \( \lambda_j \neq 0 \). WE DIFFERENTIATE (4) WRT \( S \) AND EVALUATE AT \( S = S_0 \) WHERE \( \sigma = 0 \) AND \( \lambda_j = 0 \). AT FOLD POINT WE HAVE \( \phi = \bar{U}_S \).

DIFFERENTIATING USING PRODUCT RULE

\[
\sigma' \int_{\Omega} \bar{U}_S \phi \, dx + \sigma \frac{d}{dS} \int_{\Omega} \bar{U}_j \phi \, dx = \lambda_{jj} \int_{\Omega} \phi f'(\Omega) \, dx + \lambda_j \int_{\Omega} \phi f''(\Omega) \, dx
\]

EVALUATE AT \( S = S_0 \) WHERE \( \phi = \bar{U}_S \) AND \( \lambda_j = \sigma = 0 \) WE GET

\[
\sigma'(S_0) \int_{\Omega} (\bar{U}_S)^2 \, dx = \lambda_{jj}(S_0) \int_{\Omega} \bar{U}_S f'(\Omega) \, dx
\]

\( \star \)

**Remark**: We can obtain (\( \star \)) by using solvability conditions.

We have

\[
\Delta \bar{U}_S - \lambda f'(\Omega) \bar{U}_S = \lambda_j f'(\Omega) \bar{U}_j + \lambda f''(\Omega) \bar{U}_S \bar{U}_j + \sigma \bar{U}_S + \sigma' \bar{U}_j
\]

Now at \( S = S_0 \) \( \rightarrow \) \( \Delta \bar{U}_S - \lambda f'(\Omega) \bar{U}_S = \lambda f''(\Omega) \bar{U}_S^2 + \sigma' \bar{U}_j \) since \( \bar{U} = \bar{U}_S \).

Solvability yields

\[
\lambda \int_{\Omega} f''(\Omega) \bar{U}_S^3 \, dx = -\sigma' \int_{\Omega} \bar{U}_S^2 \, dx
\]

(1)

Now

\[
\lambda \bar{w} = \lambda f'(\Omega) \bar{w} = \lambda_j f'(\Omega) \bar{w}_j
\]

with \( \bar{w} = \bar{U}_j \).

Differentiate w.r.t. \( S \) \( \rightarrow \) \( \Delta \bar{w}_S - \lambda f'(\Omega) \bar{w}_S = \lambda_j f'(\Omega) \bar{w}_j + \lambda f''(\Omega) \bar{U}_S \bar{w}_j + \lambda f''(\Omega) \bar{U}_j \bar{w}_j + \lambda f''(\Omega) \bar{U}_j \bar{U}_j \bar{w}_j.

At \( S = S_0 \), we have

\[
\Delta \bar{w}_S - \lambda f'(\Omega) \bar{w}_S = \lambda f''(\Omega) \bar{U}_S^2 + \lambda \bar{U}_j f'(\Omega)
\]

Solvability yields

\[
\lambda \int_{\Omega} f''(\Omega) \bar{U}_S^3 \, dx = -\lambda_{jj} \int_{\Omega} f'(\Omega) \bar{U}_S \bar{U}_j \, dx
\]

(2)

Now combining, we obtain from (1) and (2), that

\[
\lambda_{jj} \int_{S_0} f'(\Omega) \bar{U}_S \, dx = \sigma' \int_{\Omega} (\bar{U}_S)^2 \, dx.
\]

We want to find the sign of \( \sigma' \). This is same as (\( \star \)).
Now we write
\[ \lambda_{sj} \frac{d}{ds} \left( \int_{\Omega} \mathcal{F}(u) \, dx \right) = -\sigma' \int_{\Omega} (\nabla \bar{u})^2 \, dx \]

where
\[ \mathcal{F}(u) = \int_{\Omega} F(\nabla u) \, d\lambda = -\int_{\Omega} \frac{1}{(1+\bar{u})^2} \, d\lambda = \frac{1}{1+\bar{u}}. \quad (3) \]

Thus,
\[ \lambda_{sj} \frac{d}{ds} \left( \int_{\Omega} \frac{1}{(1+\bar{u})} \, dx \right) = -\sigma' \int_{\Omega} (\nabla \bar{u})^2 \, dx \quad \text{at} \quad S = S_0. \quad (4) \]

This is a key formula which we use as follows:

Now our PDE is the Euler-Lagrange equation of the functional \( I(u, \lambda) \) defined by
\[ I(u, \lambda) = \int_{\Omega} \left( \frac{1}{2} \left| \nabla u \right|^2 - \frac{\lambda}{1+\bar{u}} \right) \, d\bar{u}. \]

The Euler equation is
\[ \frac{\partial}{\partial x} \left( \lambda \frac{\partial}{\partial x} \right) - \frac{\partial}{\partial y} \left( \lambda \frac{\partial}{\partial y} \right) G_{ux} - \frac{\partial}{\partial x} G_{uy} = 0. \]
\[ G_{ux} = \frac{\lambda}{(1+\bar{u})^2}, \quad G_{uy} = \frac{\lambda}{(1+\bar{u})^2}. \]

So we obtain that \( u \) satisfies
\[ \frac{\lambda}{(1+\bar{u})^2} = \frac{1}{(1+\bar{u})} \cdot \lambda \]

Now
\[ -\frac{\lambda}{(1+\bar{u})^2} = \int_{\Omega} \frac{1}{(1+\bar{u})} \, dx \quad \text{evaluated on solution branch}. \]

Thus introduce a variable \( S \) by
\[ S = \int_{\Omega} \frac{1}{(1+\bar{u})} \, dx. \]

where
\[ \Delta \bar{u} = \frac{\lambda}{(1+\bar{u})^2}, \quad \bar{u} = 0 \quad \text{on} \quad \partial \Omega. \]

Then (4) becomes
\[ \lambda_{sj} = -\sigma' \int_{\Omega} (\nabla \bar{u})^2 \, dx. \quad (5) \]
The idea then is to plot the "preferred" bifurcation diagram of 
\[ S = \int_{\Omega} \frac{1}{\sqrt{1 + \gamma u}} \, dx \text{ versus } \lambda. \]

Suppose we get a picture as shown:

![Diagram showing bifurcation diagram with curves and labels](image)

\[ \lambda'(S_0) = 0, \quad \lambda_{SS}(S_0) < 0 \]

Using (5) we can obtain that \( \sigma'(S_0) > 0 \) so \( \sigma > 0 \) if \( S > S_0 \) (locally), \( \sigma < 0 \) if \( S < S_0 \) locally.

Conclusion on upper branch we have that linearization has an unstable eigenvalue since \( \sigma'(S_0) > 0 \rightarrow \sigma > 0 \) for \( 0 < S - S_0 < 1 \).

In other cases we perhaps have to take a different parameterization, and not \( S = \int_{\Omega} \frac{1}{\sqrt{1 + \gamma u}} \, dx \) as the curve can be as shown:

![Diagram showing alternative parameterization](image)

We will now implement this for the radially symmetric MEMS PDE where we can use the scale invariance technique to compute \[ \int_{\Omega} \frac{1}{\sqrt{1 + \gamma u}} \, dx = \int_{\frac{r}{(1 + \gamma) \, dr}} \text{ versus } \lambda. \]
LINEAR STABILITY: RADially SYMMETRIC

We now consider the steady-state problem

\[ U'' + \frac{1}{\gamma} \frac{U'}{U} = A \left( 1 + \frac{U}{U} \right)^2, \quad U'(0) = 0, \quad U(1) = 1. \]

**"PREFERRED"**

We want to plot \( I = \int_0^1 \frac{1}{\gamma} \, dp \) versus \( J \). This diagram has stability information. Recall our scale invariance technique:

\[ \nu(r) = -1 + \nu \left( \gamma r \right). \]

Then with \( \alpha = \nu(w) \) we get

\[ \lambda = \frac{\nu^2}{[w(\nu)]^3}, \quad \rho(0) = 1 - \frac{1}{\nu(\nu)}. \]

Where \( w'' + \frac{1}{\nu} \frac{w'}{w} = \frac{1}{\nu} \) with \( w(0) = 1, w'(0) = 0. \)

Now we calculate

\[ I = \int_0^1 \frac{\gamma (\nu r) (\nu r)}{\alpha [w(\nu)]^2} \, dp = \frac{\nu(w)}{\nu^2} \left[ \frac{\nu}{w(\nu)} \right] \right|_0^1 \frac{\nu}{w(\nu)} \] \[ J = \frac{\nu}{\nu^2} \int_0^1 \frac{\nu}{w(\nu)} \, dp = w(\nu) \int_{\nu}^1 w(\nu) \, dp. \]

**NUMERICAL IMPLEMENTATION**

Defining \( w_1, w_2, \) then we have the first order system:

\[
\begin{align*}
\frac{dw_1}{dp} &= w_2, \quad w_1(0) = 1, \quad \text{then we output at "time" } p = \gamma \\
\frac{dw_2}{dp} &= -\frac{1}{\nu} \frac{w_2}{w_1} + \frac{1}{\nu} \frac{w_1}{w_1}, \quad w_2(0) = 0, \\
J' &= \frac{p}{w_1(p)}, \quad J(0) = 0.
\end{align*}
\]

Remark to avoid the singularity near \( p = 0 \) we use \( w = 1 + \frac{p^2}{4}, w' = \frac{p}{2}, \) \( J \sim p^2/2 \). For \( p < 1 \), set \( p = \varepsilon 10^{-5} \) and put

\[
\begin{align*}
w_1(\varepsilon) &= 1 + \varepsilon^2/4, \\
w_2(\varepsilon) &= \varepsilon/2, \\
J(\varepsilon) &= \varepsilon^2/2.
\end{align*}
\]

As the initial conditions.
We then plot \( \lambda = \int_0^1 \frac{r}{(1+u)} \, dr \) versus \( \lambda \) and obtain the plot as shown below. Recall that in terms of a "parametrization" \( S \), we have
\[
\frac{\lambda_S}{dS} \left( \int_0^1 \frac{r}{(1+u)} \, dr \right) = -\frac{d}{d\lambda} \left( \int_0^1 \frac{r}{(1+u)} \, dr \right) = \sigma' \left( \int_0^1 \frac{r}{(1+u)} \, dr \right)^2 \int_0^1 r \, dr. \quad (\mathcal{W})
\]

We can choose \( S = \gamma \) from the scale invariance ODE system:

\[\int r \frac{1}{(1+u)} \, dr = \lambda.\]

- At point (I) we have \( \lambda_{\gamma \gamma} < 0 \) and \( \frac{d}{d\gamma} \left( \int_0^1 \frac{r \, dr}{(1+u)} \right) > 0 \) so \( \gamma' > 0 \) (thus slightly above fold point we have instability since \( \exists \sigma > 0 \)).
- At point (II) we have \( \lambda_{\gamma \gamma} > 0 \) and \( \frac{d}{d\gamma} \left( \int_0^1 \frac{r \, dr}{(1+u)} \right) < 0 \) so \( \gamma' < 0 \) and so on other branch slightly below the fold we have \( \sigma > 0 \rightarrow \text{unstable} \).
RECALL THAT  \[
\sigma \int_{\Omega} \bar{u}_s \phi \, d\mathbf{x} = \Lambda_s \int_{\Omega} \phi \, f(u) \, d\mathbf{x} \quad \text{AND} \quad \Delta \phi - \Lambda \frac{\partial f(u)}{\partial u} \phi = \sigma \phi
\]

SO IN POLAR COORDINATES IF \( \Omega \) IS UNIT DISK, WE HAVE
\[
\sigma \int_0^1 \bar{u}_s \phi \, r \, dr = \Lambda_s \int_0^1 \phi \, f(u) \, r \, dr.
\]
WHEN \( \Lambda_s = 0 \), \( \phi = \bar{u}_s \) AND \( \phi = \bar{u}_s \).

WE CONCLUDE ON LOWER BRANCH WHERE \( \Lambda_s > 0 \) THAT SINCE \( \bar{u}_s < 0 \rightarrow \sigma, < 0 \).

AT FIRST FOLD POINT, THE PRINCIPAL EIGENVALUE CROSSES THROUGH ZERO.

AT SECOND FOLD POINT THE SECOND EIGENVALUE CROSSES THROUGH ZERO.

THE \( \Lambda_s \) RELATION \( \bar{u}_s \) HAS ONE-ZERO CROSSING \( \rightarrow \phi = \bar{u}_s \).
Now we derive a numerical approach to calculate $\tilde{U}_S(\gamma)$

When $\tilde{U}(0) = -S$,

We recall $\tilde{U}(0) = -1 + \frac{1}{w(\gamma)}$ so $1 - S = \frac{1}{w(\gamma)}$ or $w(\gamma) = (1 - S)^{-1}$.

Now $\tilde{U}(\gamma) = -1 + w(\gamma)(1 - S)$ and $\Lambda = y^2 (1 - S)^3$.

We now calculate $w'(\gamma) y_S = (1 - S)^2$ so $y_S = \frac{(1 - S)^2}{w'(\gamma)} = \frac{[w(\gamma)]^2}{w'(\gamma)}$.

We obtain that $\tilde{U}_S = w'(\gamma) y_S (1 - S) = w(\gamma)$.

Substituting we obtain

$$\tilde{U}_S(\gamma) = \frac{w(\gamma) w'(\gamma) - w(\gamma) y_S}{w'(\gamma)}$$

on $0 < \gamma < 1$,

$$\Lambda_S = -3 (1 - S)^2 y^2 + 2 y y_S (1 - S)^3 = y (1 - S)^2 [2 y_S (1 - S) - 3 y]$$

so

$$\Lambda_S = y (1 - S)^2 \left[2 (1 - S) \frac{[w(\gamma)]^2}{w'(\gamma)} - 3 y\right] \quad (2)$$

We want to show that $\tilde{U}_S(\gamma) < 0$ on $0 < \gamma < 1$. Now we eliminate $(1 - S)$ to obtain:

$$\tilde{U}_S(\gamma) = \frac{w(\gamma) w'(\gamma) - w(\gamma) y_S}{w'(\gamma)}$$

$$\Lambda = \frac{y^2}{[w(\gamma)]^3}$$

$$\Lambda_S = \frac{y}{[w(\gamma)]^2} \left[2 \frac{w(\gamma)}{w'(\gamma)} - 3 y\right]$$

$$S = 1 - \frac{1}{w(\gamma)}$$

We want to plot $\Lambda_S$ and $\tilde{U}_S(\gamma)$ on $[0,1]$. 
TOUCHDOWN BEHAVIOR

DETAINED LOCAL ANALYSIS NEAR SINGULARITY

WHAT IS THE LOCAL BEHAVIOR NEAR THE SINGULARITY FOR

\[ u_t = u_{xx} - \frac{A}{(1 + u)^2} \quad u(\pm \frac{1}{2}, t) = 0, \quad u(x, 0) = 0. \]

WE LET \( W = 1 + u \) SO THAT

\[ W_t = W_{xx} - \frac{A}{W^2} \]

\( W = 1 \) ON \( X = \pm \frac{1}{2} \)

\( W = 1 \) AT \( T = 0 \)

NOW WE SCALE BY \( W = L^\frac{1}{2} \), \( X = X^\frac{1}{2} \), \( T = T^\frac{1}{2} \) WHERE \( T_x \) IS THE TOUCHDOWN TIME. THIS YIELDS THAT

\[ \frac{L}{T} \hat{W}_t = \frac{L}{X^2} \hat{W}_{XX} - \frac{A}{L^2 \hat{W}^2}. \]

TO BALANCE ALL OF THE TERMS \( \rightarrow \hat{X} = T^\frac{1}{2}, \quad L = T^\frac{1}{3} \). THIS SUGGESTS THAT WE SHOULD LOOK FOR A QUASI-SIMILARITY SOLUTION OF THE FORM:

\[ W(x, t) = (T_x - t)^\frac{1}{3} H \left( \frac{x}{\sqrt{T_x - t}} \right)^S \]

WHERE \( S = -\frac{1}{3} \log (T_x - t) \).

IN OTHER WORDS WE ASSUME THAT TOUCHDOWN OCCURS NEAR \( X = 0 \).

WE CALCULATE

\[ W_{xx} = x^{-\frac{2}{3}} H \hat{X}^2, \quad W_t = -\frac{1}{3} x^{-\frac{2}{3}} H + x^{\frac{1}{3}} \left[ \frac{1}{2} x^{-\frac{2}{3}} H_A + \frac{H_S}{x} \right] \]

\( X = (T_x - t) \).

THEREFORE,

\[ -\frac{1}{3} x^{-\frac{2}{3}} H + x^{-\frac{2}{3}} \left[ \frac{A}{2} H_A + H_S \right] = x^{-\frac{2}{3}} H \hat{X}^2 - x^{-\frac{2}{3}} H^{-2}. \]

THIS YIELDS THE FOLLOWING PDE FOR \( H(A; S) \)

\[ \begin{cases} 
H_S = H_{AA} - \frac{A}{2} H_A + \frac{1}{3} H - \frac{A}{H^2}, & 0 < A < \infty, \quad 0 < S < \infty, \\
H_A = 0, & \text{on } A = 0 \quad (\text{i.e. } H \text{ is even in } X). 
\end{cases} \]
For a fixed \( s \), we look for the asymptotic behavior as \( s \to +\infty \). We substitute

\[ H \sim A \Lambda^p + \ldots \]

into the equation to obtain

\[ p(p-1) A \Lambda^{p-2} - \frac{p A \Lambda^p}{2} + \frac{1}{J} \Lambda A^p - \frac{A^2 \Lambda^{-2p}}{A^2} = 0. \]

Therefore, \( p = \frac{2}{3} \) and \( H \sim A \Lambda^{2/3} A^J \Lambda \to +\infty \). However, we need the behavior for fixed \( A \) as \( s \to +\infty \).

Now to determine the behavior near the "blow-up" time.

We let \( t \to T^- \) so that \( s \to +\infty \). For a fixed \( \Lambda \) we ask what is the limiting behavior as \( s \to +\infty \).

The conjecture is that \( H(s, \Lambda) \to H_e \) as \( s \to +\infty \) on \( \Lambda \) fixed, where \( H_e \) is the equilibrium solution

\[ \frac{1}{3} H_e = \frac{\Lambda}{H_e^2} \quad \text{or} \quad H_e^3 = 3\Lambda \quad \text{or} \quad H_e = 3^{1/3} \Lambda. \]

This leads to the statement that

\[ W(x, t) \to (T_- - t)^{1/3} (3 \Lambda)^{1/3} \quad \text{as} \quad t \to T^- \quad \text{for} \quad |x| \leq C (T_- - t)^{1/2}. \]

This turns out to be correct. The following is a rigorous result:

**Proposition:** Consider \( W_t = W_{xx} - \frac{1}{W^2} \) on \( |x| \leq L, \quad t > 0, \quad B > 1 \)

with \( W(\pm L, t) = 1, \quad W(x, 0) = W_0(x) > 0 \) with \( W_0(\pm L) = 1 \). Let \( W \)

be a solution that quenches at a finite time \( T^- \), and let \( X_0 \) be the quenching point. Then,

\[ \lim_{t \to T^-} W(x, t) (T^- - t)^{-1/2(2+B)} = (B+1)^{1/2-B} \quad \text{for} \quad |x - X_0| \leq C (T^- - t)^{1/2}. \]

**Remark:** For our problem \( B = 2 \) and \( \Lambda = 1 \) so \( H_{eq} = 3^{1/3} \) which agrees with the result above.
**Numerical Results Showing Blow-up Singularity.**

\[ \lambda = 1 \]

\[ \lambda = 2 \text{ (no steady-state)} \]

**Local Behavior**

To be analyzed

**Convergence to Steady-State**

For \( \lambda = 1.4 \)

**No Steady-State Solution**

For \( \lambda = 2 \)

**Question:** What is the local behavior near the touchdown time and the touchdown point?
WE CONSIDER THE PDE
\[ W_t = W_{xx} - \frac{\Lambda}{W^3}, \quad -\frac{1}{2} < x < \frac{1}{2} \]
\[ W = 1 \text{ on } x = \pm \frac{1}{2}, \quad W = 1 \text{ at } t = 0 \]

WE LOOK FOR A QUASI-SIMILARITY SOLUTION OF THE FORM
\[ W(x, t) = (T_x - t)^{1/3} H \left[ \frac{x}{\sqrt{T_x - t}} \right] \]
\[ S = -\log (T_x - t) \]

WE LET \( \Lambda = \frac{x}{\sqrt{T_x - t}} \) AND OBTAIN
\[
\begin{cases}
H_S = H_{\Lambda\Lambda} - \frac{\Lambda}{2} H_{\Lambda} + \frac{1}{3} H - \frac{\Lambda}{H^2}, & 0 < \Lambda < \infty, \quad 0 < S < \infty \\
H_{\Lambda} = 0 & \text{on } \Lambda = 0
\end{cases}
\]

WE WANT TO DETERMINE THE LONG-TIME BEHAVIOR AS \( S \to \infty \).

DEFINE \( g(H) = \frac{H}{3} - \frac{\Lambda}{H^2} \) AND NOTICE THAT \( g(H_e) = 0 \)

WHERE \( H_e = (3\Lambda)^{1/3} \). THEN \( V = H - H_e \)
\[ g(H) \sim g'(H_e) V + \frac{g''(H_e)}{2} V^2 + \ldots \]
\[ g'(H_e) = \frac{1}{3} + \frac{2\Lambda}{H_e^3} \]
\[ g''(H_e) = -\frac{6\Lambda}{H_e^4} \]

SO THAT
\[ V_S = V_{\Lambda\Lambda} - \frac{\Lambda}{2} V_{\Lambda} + V - \frac{3\Lambda}{H_e^4} V^2 + \ldots \]

WE THEN WRITE THIS PROBLEM IN THE FORM
\[
V_5 = \frac{1}{2} V - w \, V^2 \quad w = \frac{3A}{H e^4} = \frac{3A}{(3A)^{4/3}} = \frac{1}{(3A)^{1/3}}
\]

where
\[
\frac{1}{\rho} \frac{d}{d\lambda} (\rho \, V \lambda) + V = p = e^{-\lambda^2/4}
\]

This is Hermite's operator.

We first consider the eigenvalue problem

\[
\begin{align*}
\dot{\phi} + \lambda \phi &= 0, \quad -\infty < \lambda < \infty \\
\int_{-\infty}^{\infty} p \, \phi^2 \, d\lambda &< \infty
\end{align*}
\]

This problem is \(\hat{\phi}_{\lambda\lambda} - \frac{\lambda}{2} \phi_{\lambda} + \phi = \lambda \phi\), and admits the spectrum

(i) \(\Lambda_1^+ = 1, \quad \Lambda_2^+ = \frac{1}{2} \Rightarrow \phi_1^+ = \phi_1^+ + \phi = \lambda \phi\).

(ii) \(\Lambda_1^- = 0 \Rightarrow \phi_1^- = C_1 \left( \frac{\Lambda_1}{2} \right)^{1/2} \Lambda_1\).

(iii) \(\Lambda_j^+ = 1 - \frac{j+2}{2}, \quad j = 1, 2, \ldots \Rightarrow \phi_j^+ = C_j H_{j+1}^+ \left( \frac{\Lambda_1}{2} \right)^{1/2} \Lambda_1\).

where \(H_j(x) = (-1)^j \, e^{x^2} \frac{d^j}{dx^j} \left( e^{-x^2} \right)\) is the Hermite polynomial.

Normalization of \(\int_{-\infty}^{\infty} p(\phi_1^+)^2 \, d\lambda = \int_{-\infty}^{\infty} e^{-\lambda^2/4} (\phi_1^-)^2 \, d\lambda = 1\) yields

\[
C_1^0 = \frac{1}{2 \pi^{1/4}}.
\]
Now we expand

\[ V = B_1 \phi_1^+ (\Lambda) + B_2 \phi_2^+ (\Lambda) + a_{ij} \phi_i^0 (\Lambda) + \ldots \]

\[ \begin{array}{c c c c}
\text{growing} & \text{neutral} & \text{decaying} \\
\end{array} \]

We want \( V \to 0 \), \( J \to \infty \). Hence \( B_1 = B_2 = 0 \).

We substitute into

\[ V_S = \mathcal{A} V - W V^1 \]

to obtain,

\[ a_1 \phi_1^0 = -w a_1 (\phi_1^0)^2 + \text{decaying modes.} \]

We then use orthogonality of eigenfunctions to obtain

\[ a_1 \int_{-\infty}^{\infty} (\phi_1^0)^2 \rho \, d\Lambda = -w a_1^2 \int_{-\infty}^{\infty} (\phi_1^0)^3 \rho \, d\Lambda \]

We calculate

\[ \int_{-\infty}^{\infty} \rho (\phi_1^0)^2 \, d\Lambda = 1 \text{ normalization} \]

\[ \int_{-\infty}^{\infty} \rho (\phi_1^0)^3 \, d\Lambda = \int_{-\infty}^{\infty} \rho \left( \frac{\Lambda^2}{2} - 1 \right)^3 \, d\Lambda = 4 c_1^0. \]

Hence

\[ a_1 = -4 w c_1^0 a_1^2 \]

Thus

\[ a_1 \sim A / S \quad \text{with} \quad A = \frac{1}{4 w c_1^0} \quad \text{as} \quad S \to \infty. \]

We conclude that

\[ H = H_e + \frac{A}{S} c_1^0 \left( \frac{\Lambda^2}{2} - 1 \right) + \ldots \quad \text{as} \quad S \to \infty \]

\[ c_1^0 A / S = \frac{1}{4 w S}. \]
Therefore,

\[ H \sim H_e + \frac{1}{4\sqrt{\pi}} \left( \frac{\Lambda^2}{2} - 1 \right) \quad \text{as} \quad s \to \infty, \quad \Lambda \text{ fixed.} \]

However, \( H_e = (3 \Lambda)^{1/3} \), \( W^2 = (3 \Lambda)^{1/3} \), \( \Lambda = \frac{X}{\sqrt{T^x - t}} \), \( s = 1/\log(T^x - t) \).

Then \( W \sim (T^x - t)^{1/3} H \) yields,

\[ W \sim [3 \Lambda (T^x - t)]^{1/3} \left( 1 + \frac{X^2}{\theta(T^x - t) \log |T^x - t|} - \frac{1}{4 \log |T^x - t|} \right) \]

As \( T^x - t \to 0 \) for \( \frac{X^2}{T^x - t} \) fixed, and \( W = -1 + W \).

This is the quench profile.