We will consider ODE models with a parameter $\lambda$ of the form
\[(x) \quad \frac{dx}{dt} = f(x, \lambda)\]

where for some values of $\lambda$ there is a bifurcation point of equilibria for $(x)$. We then ask how does the behavior of solution change near the bifurcation point if $\lambda$ crosses slowly through the bifurcation point in time. For instance, consider two scenarios:

\[\text{Fold point: } \lambda \text{ fixed in time} \quad \Rightarrow \quad \lambda = \lambda(\epsilon t) \quad (\epsilon \ll 1)\]

\[\text{Pitchfork bifurcation: } \lambda \text{ fixed} \quad \Rightarrow \quad \lambda = \lambda(\epsilon t) \quad (\epsilon \ll 1)\]

We will show that a slow ramping through the bifurcation point leads to a "delay" in the bifurcation. The qualitative change in the solution occurs not at the static bifurcation point but instead at some point beyond the static value.

We will analyze different types of bifurcation with this slow sweep:

(i) transcritical
(ii) pitchfork
(iii) fold point
(iv) Hopf bifurcation
TRANSITCRITICAL BIFURCATION

Consider \( X' = X \left[ \Lambda - X \right] \).

If \( \Lambda \) is a constant parameter, we get

\[
\Lambda = 0 \text{ is transcritical bifurcation point.}
\]

We now let \( \Delta = \Delta(\tau) \) with \( \Delta(0) = \Lambda; < 0 \). Let \( \tau = \tau \) and \( \dot{X} = dX/d\tau \).

We get

\[
\varepsilon \dot{X} = X \left[ \Lambda(\tau) - X \right], \quad \text{with} \quad X(0) = X_0 \neq 0
\]

Now put \( X = 1/U \) so that \( -\varepsilon \dot{U} = \frac{A}{U} - \frac{1}{U^2} \). This yields

\[
\varepsilon \dot{U} + \Lambda U = 1, \quad \Rightarrow \quad \dot{U} + \frac{\Lambda(\tau)}{\varepsilon} U = \frac{1}{\varepsilon} \quad \text{with} \quad U(0) = \frac{1}{X_0}
\]

Now multiply by the integrating factor \( \Phi = e^{\int_0^{\tau} \Lambda(s) ds} = \Phi(\tau) \)

\[
\left( \frac{1}{\varepsilon} \int_0^{\tau} \Lambda(s) ds \right) = \frac{1}{\varepsilon} \int_0^{\tau} \Lambda(s) ds
\]

We get

\[
\Phi(\tau) \cdot U = \frac{1}{\varepsilon} \int_0^{\tau} \Phi(\mu) d\mu + \frac{1}{X_0}
\]

This yields

\[
U = \frac{1}{\Phi(\tau)} \left[ \frac{1}{\varepsilon} \int_0^{\tau} \Phi(\mu) d\mu + \frac{1}{X_0} \right]
\]

or

\[
X = \frac{X_0 \Phi(\tau)}{\varepsilon \int_0^{\tau} \Phi(\mu) d\mu + 1}
\]

This is the exact solution. What is its behavior for \( \varepsilon \ll 1 \) and \( \Lambda(0) = \Lambda; < 0 \)?

We suppose that \( \Lambda(\tau) = \Lambda; + \tau \) so that \( \Lambda \) is a linear ramp that crosses through the bifurcation point at time \( \tau = -\Lambda; \).
Now we define \( \Phi(\tau) = \int_0^\tau A(s) \, ds \), so that \( \Phi(\tau) = \exp \left[ \frac{1}{\varepsilon} \Phi(\tau) \right] \).

We observe that if \( \Phi(\tau) < 0 \longrightarrow \Phi(\tau) \) is exponentially small as \( \varepsilon \rightarrow 0^+ \), and \( \int_0^\tau \Phi(\lambda) \, d\lambda \) is exponentially small if \( \Phi(\lambda) < 0 \) on \( 0 < \lambda < \tau \).

Now \( \Phi(\tau) = \int_0^\tau \frac{A(s)}{A(0)} \, ds \, d\lambda = \frac{A(\tau)}{A(0)} \int_{A(0)}^{A(\tau)} \lambda \, d\lambda = \int_{A(0)}^{A(\tau)} \lambda \, d\lambda \).

We claim that \( \Phi(\tau) < 0 \) for \( A(0) < \lambda < A; \quad (A; > 0) \)

i.e. \( 0 < \tau < -2A \).

This yields that \( x \equiv 0 \) for \( 0 < \tau < -2A \). (Stage 2).

Now suppose \( \tau > -2A \), or \( A(\tau) > -A \). Then \( \Phi(\tau) > 0 \) and so \( \Phi(\tau) \) is exponentially large.

By Laplace's method:
\[
\int_0^\tau \Phi(\lambda) \, d\lambda = \int_0^\tau \exp \left[ \frac{1}{\varepsilon} \int_0^\lambda A(s) \, ds \right] \, d\lambda.
\]

The dominant contribution to the integral comes from upper endpoint. Recalling that for \( x \rightarrow +\infty \)
\[
\int_a^b \frac{A(\lambda)}{x F(\lambda)} \, d\lambda \sim \frac{A(\lambda)}{x F(\lambda)} \left[ \frac{1}{x^2} \right] \quad \text{as} \quad x \rightarrow +\infty
\]

when \( F \) has its maximum at \( x = b \). We have that
\[
\int_0^\tau \Phi(\lambda) \, d\lambda = \int_0^\tau \exp \left[ \frac{1}{\varepsilon} \int_0^\lambda A(s) \, ds \right] \, d\lambda \sim \frac{\varepsilon}{A(\tau)} \exp \left[ \frac{1}{\varepsilon} \int_0^\tau A(s) \, ds \right] \quad \text{as} \quad \varepsilon \rightarrow 0 \quad A(\tau) \rightarrow \frac{\varepsilon}{A(\tau)} \quad \Phi(\tau) \quad (2)
\]
Therefore we have from (1) that for $\vartheta > 0$

$$x = \frac{x_0 e^{\vartheta}}{x_0 e^{\vartheta} + 1}$$

(3)

Hence if $\vartheta(\tau)$ is exponentially large, i.e., if $\lambda > \lambda_1$, we have that (3) becomes $x = \lambda(\tau)$.

This leads to the following picture for a slow ramp through the bifurcation point.

Notice that for $\varepsilon << 1$ the jump occurs at $\lambda = -\lambda_1 > 0$ which is at $\vartheta = -2\lambda_1 / \varepsilon$ or $\tau = -2\lambda_1 / \varepsilon$ and not at “static” bifurcation time $\tau_s = -\lambda_1 / \varepsilon$. There is a finite $\varepsilon$-independent delay in the value of $\lambda$ at which the jump occurs.

Remark: The key quantity which monitors whether $\vartheta(\tau)$ is exponentially small or large as $\varepsilon \to 0$ is the term

$$\int_0^\tau \lambda(s) ds = \int_1^{\vartheta(\tau)} \lambda d\tau$$

If $\vartheta > 0 \Rightarrow \vartheta(\tau) = \exp \left[ \frac{1}{\varepsilon} \int_0^\tau \lambda(s) ds \right]$ is exponentially large and the jump has already occurred.

The delayed bifurcation value is the value $\lambda_F = \lambda(\tau_F)$ for which $\int_{\lambda_1}^{\lambda_F} \lambda d\lambda = 0$ “equal-area” rule.
Now with this insight we revisit
\[ \dot{x} = x \left[ \lambda(t) - x \right] \quad \text{with} \quad x(0) = x_0. \]

For \( t = O(\varepsilon) \) we have a quick transition from \( x_0 \) to 0. (Stage I)

For \( t = O(1) \) we have stage II where \( x \) is near \( x = 0 \). We linearize to get
\[ \dot{\varepsilon} x = x \left[ \lambda(t) \right] \]

so that
\[ x = c \exp \left[ \frac{1}{\varepsilon} \int_0^t \lambda(s) \, ds \right]. \]

We have that \( x \ll 1 \) as \( \varepsilon \to 0^+ \) only when \( \Phi(\tau) \int_0^\tau \lambda(s) \, ds < 0 \).

The linearization is inconsistent when \( \tau > \tau_f \) where \( \Phi(\tau_f) = 0 \).

**Delayed bifurcation time** is \( \tau_f \) with \( \int_0^{\tau_f} \lambda(s) \, ds = 0 \).

Let \( \Lambda = \Lambda_i + \varepsilon \) we get \( \tau_f = -2 \Lambda_i \), with \( \Lambda_i < 0 \).

Remark: As an exactly solvable example consider
\[ \dot{x} = (-1 + \varepsilon t) x \quad \text{with} \quad x(0) = x_0 \]

where the "eigenvalue" is \( \lambda(\varepsilon t) = -1 + \varepsilon t \). We might naively expect that when \( \Lambda > 0 \), i.e., when \( t \geq \frac{1}{\varepsilon} \) that the solution begin to grow exponentially.

However the exact solution is
\[ x(t) = x_0 \exp \left[ \frac{(\varepsilon t - 1)^2 - 1}{2\varepsilon} \right] \]

Notice that for \( (\varepsilon t - 1)^2 - 1 > 0 \) the solution grows exponentially.

The onset of this is \( t = \frac{2}{\varepsilon} \) and not \( t = \frac{1}{\varepsilon} \) as naively obtained by the sign of the eigenvalue.

If \( (\varepsilon t - 1)^2 - 1 < 0 \) the solution decays exponentially. This occurs for \( O(1) \ll t < \frac{2}{\varepsilon} \).
Notice that

\[ \begin{align*}
\lambda &< 0 \text{ if } t < 1/\varepsilon \\
\lambda &> 0 \text{ if } t > 1/\varepsilon
\end{align*} \]

We then get a picture as shown.

Now from a linearized analysis can we predict the time for exponential growth? Suppose \( x < 1 \). Then

\[ x' = \lambda (t + \varepsilon) x \]

We let \( \tau = t + \varepsilon \) so that \( e^\tau x = \lambda(\tau) x \)

We solve to obtain

\[ x = x_0 \exp \left[ \frac{1}{\varepsilon} \int^\tau^1 \lambda(s) \, ds \right] \]

With \( \int^1 \lambda(s) \, ds = \lambda(s) = -1 + s \)

Now if \( \int^1 \lambda(s) \, ds = 0 \) then \( x \) will begin to grow exponentially. We calculate

\[ \int^1 (-1 + s) \, ds = -1 + \frac{1}{2} = 0 \]

so \( \tau = 2 \).

Remark. Suppose \( x' = -x \left[ x - \hat{f}(A) \right] \) with \( \hat{f}(A) = 1 - 4 \left( A - \frac{3}{2} \right)^2 \).

The equilibria are \( x_e = 0 \) and \( x_e = 1 - A \left( A - \frac{3}{2} \right)^2 \).

Q1: Suppose \( A(0) = A; \lambda < 1 \). When will the solution jump to the nontrivial branch if \( A'(\varepsilon) = A; \lambda + \varepsilon \) and \( x(0) = x_0 ? \)

Q2: Can we take \( \lambda \) sufficiently negative to "jump" across the unstable zone \( 1 < A < 2 \) with a ramp \( A = A_1 + \varepsilon t ? \)
REMAK 2 (PITCHFON BIFURCATON)

CONSIDER \( x' = A(\varepsilon t) x - x^3 \)

WITH \( A \) FIXED WE HAVE THE EQUILIBRIA \( x_0 = 0 \) AND \( x_0 = \pm \sqrt{A} \).

THIS YIELDS THE BIFURCATION DIAGRAM WITH \( A \) CONSTANT.

NOW SUPPOSE \( A(\varepsilon t) \) OSCILLATES SLOWLY

ACROSS THE PITCHFON BIFURCATION VALUE \( A = 0 \). WHAT TYPE OF DYNAMICS WILL OCCUR?

EXAMPLE CONSIDER THE ODE

\[ x' = x [-1 + A f(x)] \]

WITH \( f(0) = 1 \) AND \( x > 0 \).

SUPPOSE THAT \( A = A(\varepsilon t) \) WITH \( \varepsilon = \varepsilon t \) AND \( A(0) < 1, \varepsilon < 1 \).

(i) EQUILIBRIA ARE \( x = 0 \) AND \( A f(x_0) = 1 \) SO \( A = 1 \) IS A TRANSCRITICAL BIFURCATION POINT.

(ii) FOR \( x \geq 0, f(0) = 1 \) AND \( x' = (A-1)x \).

THUS FOR A CONSTANT AND \( A > 1 \) \( \rightarrow x = 0 \) UNSTABLE

\( A < 1 \) \( \rightarrow x = 0 \) STABLE.

NOW LET \( A(t) = A_0 + \varepsilon t \) WITH \( A_0 < 1 \) AND \( \varepsilon = \varepsilon t \). THE LINEARIZED PROBLEM IS

\[ x' = [A_0 - 1 + \varepsilon t] x \]

IS \( \frac{dx}{x} = [A_0 - 1 + \varepsilon t] dt \) SO THAT

\[ \ln \left( \frac{x}{x_0} \right) = [A_0 - 1] t + \varepsilon \frac{t^2}{2} \quad \rightarrow \quad x = x_0 \exp \left[ (A_0 - 1) + \frac{\varepsilon t^2}{2} \right] . \]

INSTABILITY OCCURS WHEN \( t = t^* \) WHERE \( A_0 - 1 + \varepsilon \frac{t^*}{2} = 0 \) SO THAT

\[ t^* = \frac{2}{\varepsilon} (1 - A_0) \quad \text{AND} \quad A(t^*) = A_0 + \varepsilon t^* = A_0 + \varepsilon \frac{2}{\varepsilon} (1 - A_0) = 2 - A_0 . \]

IF \( A_0 < 1 \), \( A_0(t^*) = 2 - A_0 > 1 \). NOTICE THAT \( A_0(t^*) \) INDEPENDENT OF \( \varepsilon \)

\[ \frac{x}{x_0} = A - 1 \]

AND \( x = x_0 \exp \left[ (A_0 - 1) + \frac{\varepsilon t^2}{2} \right] . \]

LETS \( f(x) = \frac{1}{x < 1} \) \( \rightarrow \quad \text{EQUILIBRIA} \)
We now consider the case of a **saddle-node bifurcation point**.

We consider the scalar problem

\[ y' = f(y, \lambda) \]

For instance, if \( f(y, \lambda) = y - \frac{y^3}{3} + \lambda \), equilibria are \( \lambda = y_c + y_c^3/3 \).

The plot is as shown. Now \( d\lambda/dy = 0 \) at \( y_c^3 = 1 \), so \( y_c = 1 \rightarrow \lambda = \frac{2}{3} \).

We have a fold point at \( y_c = 1, \lambda = \frac{2}{3} \) and at \( y_c = -1, \lambda = \frac{2}{3} \).

We define \( \lambda_c = \frac{2}{3}, y_c = 1 \). We suppose that \( \lambda = \lambda_c + \epsilon t \) so that \( \lambda \) slowly increases beyond the saddle-node point. When does the transition to the upper branch occur? A zoom of region near saddle-node point is

Now we let \( y = y_c + \epsilon^p y + \ldots \)

and \( t = \epsilon^q \). We substitute to get

\[ y' = f(y, \lambda) \approx f^0 + f_y^0 (y - y_c) + \frac{f_{yy}^0 (y - y_c)^2}{2} + \frac{f_A^0 (\lambda - \lambda_c)}{2} + \ldots \]

Here \( f^0 = f(y_c, \lambda_c), \ f_y^0 = f_y(y_c, \lambda_c) \).

Now at the fold point we have \( f^0 = f_y^0 = 0 \). Thus, we get

\[ y' \approx \frac{f_{yy}^0 (y - y_c)^2 + f_A^0 (\lambda - \lambda_c)}{2} + \ldots \]

Now

\[ \frac{dy}{dt} = \epsilon^{p+q} \frac{dy}{dr} = \frac{f_{yy}^0}{2} (\epsilon^{2p} y_r^2) + f_A^0 \epsilon t \]
However \( t = e^8 \) so that
\[
\epsilon^p q \frac{d}{dt} y_1 = \frac{F_{yy}}{2} \left( \frac{e^2 p y_1^2}{2} + \epsilon^{1+8} \right) = F_{\alpha}^o.
\]

To balance the terms we get
\[
p-q = 2p = 1+8 \quad \rightarrow \quad p = \frac{1}{3}, \quad q = -\frac{1}{3}
\]

Thus, we obtain
\[
y = y_c + \epsilon^{1/3} y_1 + \ldots, \quad t = \epsilon^{-1/3} \tau
\]

which yields the approximate equation
\[
(1) \quad \dot{y}_1 = \frac{F_{yy}}{2} y_1^2 + F_{\alpha}^o \tau
\]

Now derive a canonical ODE: Let \( y_1 = BV \), \( \tau = SS \), and \( V = VI(S) \).

We obtain,
\[
\frac{B}{S} \frac{d}{dS} V = \frac{F_{yy}^o B^2}{2} V^2 + F_{\alpha}^o SS \quad \text{with} \quad V = \frac{dV}{dS}
\]

Now in our example \( F_{yy}^o = -2y_c = 2 > 0 \) (since \( F = y - \frac{y^3}{3} + \lambda \)) and \( F_{\alpha}^o = 1 > 0 \).

Since \( F_{yy}^o > 0 \) and \( F_{\alpha}^o > 0 \) we write
\[
\frac{B}{S^2 F_{\alpha}^o} \frac{d}{dS} V = \frac{F_{yy}^o B^2}{2 F_{\alpha}^o S} V^2 + S
\]

We set \( \frac{B}{S^2 F_{\alpha}^o} = 1 \) and \( \frac{F_{yy}^o B^2}{2 F_{\alpha}^o S} = -1 \).

Now divide
\[
\frac{F_{yy}^o B^2}{2 F_{\alpha}^o S} = -1 \quad \Rightarrow \quad \frac{F_{yy}^o S^3 F_{\alpha}^o}{2} = -1
\]

\[
\text{Thus we have that} \quad \delta^3 = \frac{2}{F_{yy}^o F_{\alpha}^o} \quad \rightarrow \quad \delta = \left( \frac{2}{F_{yy}^o F_{\alpha}^o} \right)^{1/3}
\]
Then \[ B = f_A^0 \left( \frac{2}{f_{yy}^0 f_A^0} \right)^{2/3} \]

In summary, we get

\[ (3) \quad y = B v(s), \quad \tau = s \quad \text{with} \quad B = (f_A^0)^{1/3} \left( \frac{2}{f_{yy}^0 f_A^0} \right)^{2/3}, \quad s = - \left( \frac{2}{f_{yy}^0 f_A^0} \right)^{1/3} \]

And our ODE (2) becomes

\[ (4) \quad v' = -v^2 * s. \]

Notice that since \( s < 0 \) we have \( \tau \uparrow \rightarrow s \downarrow \). Since \( \tau > 0 \) corresponds to a slow ramp beyond the fold point, \( s \) decreasing below zero is the effect in \( s \)-plane.

Recall also that

\[ y = y_c + \varepsilon^{1/3} y_1 + \ldots \quad t = \varepsilon^{1/3} \tau \quad \text{with ramp} \quad \Lambda = \Lambda_c + \varepsilon t \]

so \( \Lambda - \Lambda_c = O(\varepsilon^{2/3}) \).

Now (4) is Ricatti's equation, it is a normal form ODE since it involves no parameters. Put \( v = \phi' \). Then \( v' = \frac{\phi''}{\phi} - \frac{\phi'^2}{\phi^2} \) so that

\[ \frac{\phi''}{\phi} - \frac{\phi'^2}{\phi^2} = \frac{\phi'^2}{\phi^2} + S \]

This yields Airy's equation

\[ (5) \quad \phi'' - S \phi = 0 \]

\( s < 0 \rightarrow \Lambda > \Lambda_c \]

\( s > 0 \rightarrow \Lambda < \Lambda_c \)

We are looking for a solution that corresponds to \( v \) bounded as \( s \rightarrow +\infty \).
The general solution is
\[ \psi = q_0 \cdot A; (s) + q, B; (s) \]
where \( A; (s) \), \( B; (s) \) are two independent solutions:

**First zero** \( A; (s_0) = 0 \rightarrow S_0 \approx 2.338 \)

Thus \[ V = \frac{q_0 \cdot A; (s) + q, B; (s)}{q_0 \cdot A; (s) + q, B; (s)} \] (6)

Notice that \( \psi \) depends only on 1-parameter, the ratio \( q_1/q_0 \).

Now for \( S \to +\infty \) we have
\[ A; (s) \sim \frac{1}{2 \sqrt{\pi}} S^{-1/4} e^{-2 S^{3/4}}, \quad B; (s) \sim \frac{1}{\sqrt{\pi}} S^{-1/4} e^{2 S^{3/4}} \] (7)

Thus if \( q, \neq 0 \) we have \( V \sim \frac{B; (s)}{A; (s)} \sim S^{1/2} \to A_j S \to +\infty \).

This would mean that
\[ Y = Y_c + e^{1/3} B V \sim Y_c + e^{1/3} B \sqrt{S} \quad \text{as} \quad S \to +\infty \]

However, we must have \( Y < Y_c \) for \( S \gg 1 \) → This is incorrect. We must have \( q_{1} = 0 \).

We set \( q_{1} = 0 \) so that \( V = \frac{A; (s)}{A; (s)} \)

Now as \( S \to +\infty \) we use (7) to obtain that \( \frac{A; (s)}{A; (s)} \sim -\sqrt{S} \) \( A_j S \to +\infty \)

which gives
\[ V \sim -\sqrt{S} \quad \text{and} \quad Y \sim Y_c - e^{1/3} B \sqrt{S} \quad \text{for} \quad S \to +\infty. \]
Now we write this in terms of the original variables.

We have \( S = \tau / \delta \) so that

\[
\gamma = \gamma_c - \epsilon^{1/3} B \sqrt{-\tau} \quad \text{with} \quad B = \sqrt{2 \frac{f_A^o}{f_{yy}^o}}
\]

with \( t = \epsilon^{1/3} \tau \) we get

\[
\gamma = \gamma_c - \epsilon^{1/3} \sqrt{2 \frac{f_A^o}{f_{yy}^o}} \sqrt{-t} \quad \gamma_c - \frac{2 f_A^o}{f_{yy}^o} (-\epsilon t)^{1/2} \quad \text{for} \; t < 0.
\]

This is the expression just before the fold point.

Now we have, returning to (6) with \( q_1 = 0 \) that

\[
\gamma = A_1 (s) / A_1 (s)
\]

we then have

\[
\gamma = \gamma_c + \epsilon^{1/3} B \sqrt{s}
\]

We conclude that \( \gamma \) becomes infinite, i.e. \( \gamma - \gamma_c \) is no longer small when \( S = S_0 \) where \( A_1 (s_0) = 0 \), i.e. the first zero of the Airy function. Thus value is \( S_0 = -2.33 \) and \( A \to + \infty \) as \( S \to S_0^+ \).

Returning to the original variables we have that the approximation breaks down

\[
t = t_B = \epsilon^{1/3} \tau_B = \epsilon^{1/3} \delta S_0 \quad \text{with} \quad \delta = - \left( \frac{2}{f_{yy}^o f_A^o} \right)^{1/3}.
\]

This yields

\[
\Lambda = \Lambda_c + \epsilon t \quad \text{or at} \; t = t_B, \quad \text{for which} \; \Lambda = \Lambda_B, \quad \text{given by}
\]

\[
\Lambda_B = \Lambda_c + \epsilon^{2/3} \tau_B = \Lambda_c + \epsilon^{2/3} (2.338...) \left( \frac{2}{f_{yy}^o f_A^o} \right)^{1/3}
\]
WE CONCLUDE WITH A SLOW RAMP $\Lambda = \Lambda_c + \varepsilon t$
that the "jump" does not occur at $t = 0$ but instead at
a scaling law with $t = O(\varepsilon^{-1/3})$ past $t = 0$.

**Example** consider $y' = y - y^{3/3} + \Lambda$, where $f = y - y^{3/3} + \Lambda$.

At the lower fold point $f_{yy} = -2y$, $f_\Lambda = 1$ so with $y = y_c = -1$
and $\Lambda_c = 3/3$ we get $f_{yy} = 2$, $f_\Lambda = 1$.

We calculate $\delta = -\left( \frac{2}{f_{yy} f_\Lambda} \right)^{1/3} = -1$ and so

\[ t_B = \varepsilon^{-1/3} \left( 2.33811 \ldots \right) \]
\[ \Lambda = \frac{2}{3} + \varepsilon^{2/3} \left( 2.3381 \ldots \right) \]

with $\varepsilon = 0.01$, $t_B = 10.85$.

**Remark** $t_B \downarrow \iff f_{yy} \uparrow$, thus for branches with higher
curvature, i.e., $\gamma$, then blowup time occurs sooner.
Hysteresis Loop

We consider \( y' = y - \frac{1}{3} y^3 + \lambda \)

where \( \lambda = \lambda_c (1.2) \sin(\epsilon t) \) with \( \epsilon = 1 \), and \( y(0) = -1, \lambda_c = \frac{2}{3} \)

We get a hysteresis loop with delay.