In 2-D the Navier-Stokes equation are
\[
\Delta^2 \psi + \text{Re} J_r[\psi, \partial \psi] = 0 \quad J_r(a, b) = \frac{i}{r} \left[ \frac{d_r a}{d_q b} - \frac{d_q a}{d_r b} \right]
\]
(look for an even solution)

\text{Re is Reynolds number.}

\[\psi = \psi_0 = 0 \text{ on body} \]

\text{Stream Function}

Moffatt: J. Fluid Mechanics 1-18, 1964

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In a small neighbourhood of the origin, the stream function \( \psi \) satisfies
\[
\Delta^2 \psi = 0, \quad \text{also} \quad \Delta \psi = -\omega, \quad \nabla^2 \omega = 0 \quad (\omega \text{ vorticity})
\]

With boundary conditions

\[\psi = 0 \text{ at } q = \frac{1}{2} \alpha, \quad \psi_q = 0 \text{ at } q = \frac{1}{2} \alpha \quad (\text{wall is a streamline and no flow through wall})\]

the fluid velocities are \( v_r = \frac{1}{r} \frac{d\psi}{dq}, \quad v_q = -\frac{d\psi}{dq} \).

Now we let \( \psi = r^A f(q) \) where we want \( \text{Re} \lambda > 1 \) for bounded velocities.

We let \( g(q) = f''(q) + A^2 f(q) \). Then, we get
\[
\left( \frac{d_r^2 + \frac{1}{r} d_r + \frac{1}{r^2} d_q^2} r^2 \right) r^A g(q) = 0 \quad \rightarrow \quad \left( \frac{d_r^2 + \frac{1}{r} d_r + \frac{1}{r^2} d_q^2} r^2 \right) r^{A-2} g'(q) = 0
\]

We want an even solution to correspond to streamline shown above

\[g(q) = \tilde{A} \cos \left[ (A-1) q \right]
\]

then
\[f''(q) + A^2 f(q) = \tilde{A} \cos \left[ (A-2) q \right].\]

An even solution has the form for \( A \neq 1, \# 2 \) and \( A \neq 0 \). Note \( v_r \sim r^{-A} f'(1/r) \) is odd in \( q \).

\[f(q) = A_0 \cos \left( A q \right) + C \cos \left( (A-2) q \right)
\]

Hence,
\[
\psi \sim \sum_{n=1}^{\infty} \tilde{A}_n \cos \left[ A_0 q \right] + C_n \cos \left[ (A_0-2) q \right]
\]

In the corner, if \( A = 1 \) then \( f(q) = A_0 \cos q + C \cos 2q \) and \( C \) is irrelevant to the problem since we want bounded velocities, i.e. \( A > 1 \).
We remark that upon combining

\[(1) \quad g''(\phi) + (A-2)^2 g(\phi) = 0\]

with \(g = F'' + A^2 F\) we obtain the \textbf{fourth-order eigenvalue problem}

\[\begin{align*}
F'''(\phi) + A^2 F''(\phi) + (A-2)^2 F''(\phi) + A^2 (A-2)^2 F(\phi) &= 0, \\
F(\phi) &= F'(\phi) = 0, \quad F(-\phi) = F'(-\phi) = 0.
\end{align*}\]

This is not self-adjoint and there is no guarantee of real eigenvalue for this problem.

We solve \((1)\) by first finding even solutions to \((1)\) for \(g(\phi)\) and then finding solutions for \(F(\phi)\).

This gave

\[F(\phi) = A \cos(A \phi) + C \cos((A-2) \phi)\]

Now we set \(F(\phi) = F'(\phi) = 0\) (by symmetry it follows that \(F(-\phi) = F'(-\phi) = 0\)).

We obtain that

\[A \cos(A \phi) + C \cos((A-2) \phi) = 0\]
\[A \sin(A \phi) + C (A-2) \sin((A-2) \phi) = 0\]

\(A\) \text{ a nontrivial solution iff } \(A\) satisfies the eigenvalue equation

\[(2) \quad \det\begin{bmatrix} \cos(A \phi) & \cos((A-2) \phi) \\ \sin(A \phi) & (A-2) \sin((A-2) \phi) \end{bmatrix} = 0.\]

Once a root to \((2)\) is found we can set \(A = \Re (\cos((A-2) \phi))\),
\(C = -\Im (\cos(A \phi))\), so that

\[(3) \quad \psi \sim \Re A \left[ \cos(A \phi) \cos((A-2) \phi) - \cos(A \phi) \cos((A-2) \phi) \right] \]
Next, we simplify the eigenvalue relation

\[(\Lambda - 2) \sin \left( (\Lambda - 2) \alpha \right) \cos (\Lambda \alpha) = \Lambda \cos \left( (\Lambda - 2) \alpha \right) \sin (\Lambda \alpha)\]

Subtract and add 2 \( \cos \left( (\Lambda - 2) \alpha \right) \sin (\Lambda \alpha) \) to the right side to obtain

\[ (\Lambda - 2) \sin \left( (\Lambda - 2) \alpha \right) \cos (\Lambda \alpha) = (\Lambda - 2) \cos \left( (\Lambda - 2) \alpha \right) \sin (\Lambda \alpha) + 2 \cos (\Lambda - 2) \alpha) \sin (\Lambda \alpha) \]

This yields

\[ (\Lambda - 2) \left[ \sin ((\Lambda - 2) \alpha) \cos (\Lambda \alpha) - \cos ((\Lambda - 2) \alpha) \sin (\Lambda \alpha) \right] = 2 \cos ((\Lambda - 2) \alpha) \sin (\Lambda \alpha) \]

Using the trig identity given

\[ - (\Lambda - 2) \sin (2 \alpha) = 2 \cos ((\Lambda - 2) \alpha) \sin (\Lambda \alpha) \]

Next, use \( \sin (A \cos B) = \frac{1}{2} \sin (A - B) \cos (A + B) \) with \( A = \Lambda \alpha \), \( B = (\Lambda - 2) \alpha \) to get

\[ - (\Lambda - 2) \sin (2 \alpha) = 2 \left( \frac{1}{2} \right) \left[ \sin (2 \alpha) + \sin (2(\Lambda - 1) \alpha) \right] \]

or

\[ -(\Lambda - 2) \sin (2 \alpha) = \sin (2 \alpha) + \sin (2(\Lambda - 1) \alpha) \]

This finally yields that \( \Lambda \) satisfies

\[ (4) \quad -(\Lambda - 1) \sin (2 \alpha) = \sin (2(\Lambda - 1) \alpha) \]

Thus, upon writing \( \mu = \Lambda - 1 \) we obtain that \( \mu \) satisfies

\[ (5) \quad \sin (2 \mu \alpha) = -\mu \sin (2 \alpha) \]

In terms of \( \mu \) we have from (3)

\[ (6) \quad \mu \sim \sqrt{\frac{1 + \mu}{\mu}} \left[ \cos (1 + \mu) \phi \cos \left( (\mu - 1) \alpha \right) - \cos (1 + \mu) \phi \cos \left( (\mu - 1) \phi \right) \right] \]
Our goal is for fixed $\kappa$ in order to find the smallest root of (5) satisfy $\mu > 0$ so that $\mu \to 0$ as $\gamma \to 0$, and velocity $\to 0$ as $\gamma \to 0$.

Notice that $\mu = 0$ is a root $\forall \kappa$. This gives $\lambda = 1$. However, can we find a root to (5) with $\mu$ real and $\mu > 0$.

Thus we set $\gamma = 2\mu \psi$ to obtain

(7) $\sin \gamma = -\gamma B$ with $B = \sin \left( \frac{2\kappa}{\psi} \right) \frac{2 \psi}{2 \psi}$

Notice that

$B > 0 \text{ on } 0 < \kappa < \pi/2 \Rightarrow B \to 1 \text{ as } \kappa \to 0^+ \text{. Case I}$

$B < 0 \text{ on } \pi/2 < \kappa < \pi \Rightarrow B \to 0 \text{ as } \kappa \to \pi^+ \text{ and } \kappa \to -\pi$.

Notice $|B| < 1$. Case II

Now this gives the picture for $B < 0 \text{. } \pi/2 < \kappa < \pi$. Since $-1 < B < 0$ we have

$\sin \gamma$

Thus $\exists \gamma_0 \text{ in } 0 < \gamma < \pi/2 \text{ for which there is a root. This yields}$

$\mu = \gamma_0 / 2 \psi$, with $\mu > 0$ and $\gamma_0 \frac{2 \psi}{\pi} < 1$, so $\mu < 1$.

We conclude that when $\pi/2 < \kappa < \pi$, $\exists$ root $\mu_0$ real to (5) that satisfies $\mu_0 < 1$.

(1) Next, suppose $0 < \kappa < \pi/2$ so that $B > 0$ with $0 < B < 1$.

Then we have $\sin \gamma$
WE CONCLUDE THAT:

(i) If $\alpha < \alpha_C$ there are no solution with $\gamma$ real to $\sin \gamma = -B$

and hence no real $\mu$. Here $\alpha_C$ satisfy the tangency problem

\[ \sin \gamma = -B \]

\[ \cos \gamma = -B \]

which yield $\tan \gamma = \gamma$ in $\pi/2 < \gamma < 3\pi/2$ where $\sin \gamma < 0$, $\cos \gamma < 0$.

We calculate $\gamma_0 \approx 4.4934$ by Newton so $\cos \gamma_0 \approx -0.217233$.

Then $2\alpha_C$ satisfy

\[ \sin \left[ \frac{2\alpha_C}{\alpha_C} \right] = -\cos(\gamma_0) \approx 0.217233 \]

which yield

\[ 2\alpha_C = 2.5535 \text{ (radians)} \]

or (8) $\alpha_C = \frac{2.5535}{2} \left( \frac{180}{\pi} \right) \approx 73.15^\circ$. \[ \sqrt{146.3^\circ = 2\alpha_C} \]

We conclude that if $\alpha < \alpha_C$, then no real $\mu$ exist.

Remark: There are real solution if $\alpha > \alpha_C = 73.15^\circ$ with a $\mu < 0$.

And we are looking for a solution with $\mu > 0$.

We calculate $\mu = \frac{\gamma_0}{2\alpha}$ for $\alpha > \alpha_C$.

At $\alpha: \alpha_C$, $\mu = \frac{4.4934}{2.5535} \approx 1.759$.

Notice as $\alpha \rightarrow \pi/2$, $\mu \rightarrow \frac{\pi}{\pi} = 1$.

Notice that as $\alpha \rightarrow \alpha_C^+$, two nearby real roots become complex and disappear into complex plane for $\alpha < \alpha_C$. 
Now look for complex roots $\mu$ of

$$\sin(2\mu \alpha) = -\mu \sin(2\alpha)$$

when $\alpha < \alpha_c$. We want to find the root $\mu^*$ with the smallest $\text{Re}(\mu^*)$ in $\text{Re}(\mu^*) > 0$.

We let $\mu = p + iq$ and let $\gamma = 2\alpha p$, $\Lambda = 2\alpha q$.

$$\sin(\gamma + i\Lambda) = -(p + iq) \sin(2\alpha)$$

Thus,

$$\sin(\gamma + i\Lambda) = -(q + i\Lambda) \eta, \quad \eta = \frac{\sin(2\alpha)}{2\alpha}.$$

We use $\sin(\gamma + i\Lambda) = \sin \gamma \cosh \Lambda + i \cos \gamma \sinh \Lambda$.

This gives that $\gamma, \Lambda$ satisfy the coupled system in $\gamma > 0$

$$\begin{cases}
\sin \gamma \cosh \Lambda = -\eta \gamma \\
\cos \gamma \sinh \Lambda = -\eta \Lambda
\end{cases}.$$

WLOG we take $\Lambda > 0$.

We seek the root of (9) for which $\gamma > 0$ is the smallest.

In terms of the roots $\gamma_m, \Lambda_m, m = 1, 2, 3, \ldots$ we have

$$\Lambda = 1 + \frac{1}{2\alpha} (\gamma_m + i\Lambda_m), \quad m = 1, 2, 3, 4, \ldots$$

The dominant behavior occurs for $m = 1$ which gives the root for which $\text{Re}(\Lambda_m)$ is smallest.

Clearly, any root to (9) in $\eta > 0$ must occur when $\sin \gamma < 0$ and $\cos \gamma < 0$. Thus we have

$$(2m-1)\pi < \gamma_m < \frac{(2m-1)}{2} \pi.$$
In particular, we compute numerically that:

<table>
<thead>
<tr>
<th>2α</th>
<th>β</th>
<th>4</th>
<th>RE (41.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>90°</td>
<td>4.30</td>
<td>1.77</td>
<td>1 + 4.30</td>
</tr>
<tr>
<td>70°</td>
<td>4.26</td>
<td>1.97</td>
<td>3.519</td>
</tr>
<tr>
<td>140°</td>
<td>4.46</td>
<td>0.64</td>
<td>1.759</td>
</tr>
</tbody>
</table>

The key issue is what does complex \(41.\) signify?

It indicates zones of recirculation or eddies in the local flowfield, infinite sequence of closed streamline eddies in any wedge with angle \(0 < 2α < 146°\).

We write \(Ψ = Κ R^A \left[ g(q, A) \right] \)

with \(g(q, A) = \cos A q \cos (A/2 - q) - \cos A q \cos (A/2 + q)\)

for \(\Lambda = 1 + p^i q^j\) then \(R^1 + p^i q^j = R^1 + p^i \left[ \cos (q^i q^j) + i \sin (q^i q^j) \right] \)

we decompose \(g(q, 1 + p^i q^j) = g_R(q) + i g_I(q)\)

then \(\Psi_R = \text{RE} \left[ Ψ \right] = Κ R^1 \left[ \cos (q^i q^j) g_R(q) - \sin (q^i q^j) g_I(q) \right] \)

where we take \(q > 0\).

This function \(\Psi_R(q, \Phi)\) has for each \(\Phi\) in \(|\Phi| < 2α\) an infinite number of zero crossings \(A\) \(\Gamma \rightarrow 0\). They are at roots \(\Gamma_0\) of \(\tan(-q_i \Phi) = -\frac{g_I(q)}{g_R(q)}\)

Let \(X = -q_i \Phi \rightarrow +\infty \) as \(\Gamma \rightarrow 0^+\). \(\tan X = \alpha\)

so \(-q_i \Phi_n \rightarrow \tan^{-1} \left[ \frac{-g_I}{g_R} \right] + (n + \frac{1}{2})π\) \(n = 0, 1, 2, \ldots\)

Thus we have \(\Psi_R = 0\) at these infinite \# points \(\Gamma_n\) with \(\Gamma_n \rightarrow 0\) as \(n \rightarrow \infty\).

The function \(\Psi_R\) oscillates between these zeroes.
Remark to examine this further let's analyze the transverse velocity component along the midline \( q = 0 \).

\[
V_q \big|_{q = 0} = -\frac{\partial w}{\partial r} \bigg|_{q = 0}.
\]

Now

\[
\frac{\partial w}{\partial r} \bigg|_{q = 0} = \lambda \hat{q} \cdot \mathbf{g}(r, \lambda) = r^p \hat{g} \cdot \hat{g}(r, \lambda) = \lambda \hat{q} \cdot g(\lambda, \lambda)
\]

We obtain the real part

\[
\text{Re} \left[ \frac{\partial w}{\partial r} \bigg|_{q = 0} \right] = r^p \left[ (\sin(q, \lambda r) + i \sin(q, \lambda r)) \cdot \hat{g}_r + i \hat{g}_x \right] = r^p \left[ \cos(q, \lambda r) \hat{g}_r - \sin(q, \lambda r) \hat{g}_x \right] = \frac{r^p}{(\hat{g}_r^2 + \hat{g}_x^2)} \sin(q, \lambda r + B)
\]

Thus

\[
V_q \big|_{q = 0} = C r^p \sin(q, \lambda r + B) \quad \text{for some } B, C.
\]

Notice

\[
V_q \big|_{q = 0} = 0 \quad \text{when} \quad q, \lambda r + B = -\pi, \pi, 2\pi, \ldots
\]

\[
\Gamma_n = e^{-\frac{(n+1)\pi}{q_b}}\quad \text{with} \quad \frac{\Gamma_{n+1}}{\Gamma_n} = e^{-\frac{\pi}{q_b}}
\]

so \( \Gamma_n \to 0 \) as \( n \to \infty \), geometric progression.

These points are at the center of the eddies. The maxima of \( V_q \big|_{q = 0} \) occur at \( q, \lambda r + B = -(2m+1)\pi/2 \) \( m = 0, 1, 2, \ldots \) \( \Gamma_m = \frac{e^{-\frac{(2m+1)\pi}{q_b}}}{q_b} \)

\[
\left( \frac{V_q \big|_{q = 0}}{(V_q \big|_{q = 0})_m} \right)_{m+1} = \left( \frac{\Gamma_{m+1}}{\Gamma_m} \right)^p = e^{-\frac{\pi}{q_b} p}, \quad \text{thus the intensity of the eddies decrease exponentially as } \Gamma \to 0.
\]
Remark from a completely different application we can use the result to prove that the fundamental mode of vibration of a plate in a square domain is not of one sign, when the boundaries are clamped.

\[ u = \partial u = 0 \]

\[ \Delta^2 u - \lambda u = 0 \]

We have a 90° corner so that \( 2\alpha < 146.3° \Rightarrow \) Moffat eddies exist in every corner.
LAPLACE'S EQUATION: CORNER SINGULARITIES

TWO-DIMENSIONS

FOR LAPLACE'S EQUATION IN A WEDGE WE HAVE

\[ \nabla^2 u = 0 \]

WE LOOK FOR \( u = \Gamma \Lambda g(\varphi) \) AND TRY TO

CALCULATE THE SMALLEST VALUE OF \( \Lambda \) WITH \( \Lambda > 0 \). WE

NEED \( u \) BOUNDED IN THE CORNER

WE SUBSTITUTE TO GET

\[
\begin{align*}
g''(\varphi) + \Lambda^2 g(\varphi) &= 0 \\
g(0) &= g(B) = 0
\end{align*}
\]

\[ \Rightarrow g = k \sin \left( \frac{\pi \varphi}{B} \right) \quad \frac{\Lambda}{B} > 0 \]

\[ u = \text{smallest root.} \]

HENCE FOR A PROBLEM OF THE FORM

\[ \nabla^2 u = f \quad \text{WHERE } f \text{ IS SMOOTH, WE EXPECT THAT} \]

\[ u \sim k \Gamma \frac{\pi}{B} \sin \left( \frac{\pi \varphi}{B} \right) + o \left( \Gamma \frac{\pi}{B} \right) \]

\[ u \sim \text{infinite as } \Gamma \to 0. \quad (\text{CORNER REGION}) \]

CALCULATING \( k \) IS A GLOBAL PROBLEM AND CANNOT IN GENERAL

BE DONE.

1. IF \( B = 2\pi \)

\[ u \sim k \Gamma \frac{1}{2} \sin \left( \frac{\varphi}{2} \right) \]

\[ u \text{ IS INFINITE AS } \Gamma \to 0. \]

2. NOTE \( u \) IS INFINITE WHENEVER \( B > \pi \).

NOW THE QUASI-STEADY PROPAGATION OF A CRACK IS

GOVERNED BY THE FOLLOWING PROBLEM

\[ \Delta^2 \psi = 0 \quad \text{in } \Omega \setminus \Gamma, \quad \Omega \in \mathbb{R}^2 \]

\[ \psi = \psi_n = 0 \quad \text{on both sides of } \Gamma. \]

WHAT IS THE SINGULARITY IN THE CORNER REGION? TO LEADING ORDER

\[ \psi \sim A_1 \varphi^{3/2} B_1(\varphi) + A_2 \varphi^{3/2} B_2(\varphi) \quad A_i, A_2 \text{ STRESS INTENSITY FACTORS.} \]
Consider Laplace's equation in 3 dimensions:

\[ \nabla^2 V = 0 \]

Introduce spherical coordinates:

Then Laplace's equation \( \Delta V = 0 \) is:

\[ \frac{V_{rr}}{r} + \frac{2}{r} V_r + \frac{1}{r^2 \sin \theta} \frac{d}{d \theta} \left( \sin \theta \frac{dV}{d \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{d^2 V}{d \phi^2} = 0 \]

Now suppose we have azimuthal symmetry so that \( V(\rho, \phi) \)

i.e. \( 0 \leq \phi \leq B \)

we write \( V = A \rho^\mu \phi(\phi) \)

Substituting we get:

\[ \left[ \mu (\mu - 1) + 2 \mu \right] Q + \frac{1}{\sin \phi} \frac{d}{d \phi} \left( \sin \phi \frac{dQ}{d \phi} \right) = 0 \]

Therefore:

\[ \frac{1}{\sin \phi} \frac{d}{d \phi} \left( \sin \phi \frac{dQ}{d \phi} \right) + \mu (\mu + 1) Q = 0 \quad (*) \quad (\text{see below}) \]

Now let \( x = \cos \phi \), then the equation transforms to Legendre's equation:

\[ \frac{d}{dx} \left[ (1-x^2) \frac{dp}{dx} \right] + \mu (\mu + 1) p = 0 \quad \text{for } \cos B < x < 1. \]

We denote the solution by \( P_\mu(x) \).

Hence:

\[ V \sim A \rho^\mu P_\mu(\cos \phi) \]

where \( \mu \) is the smallest value with \( \mu > 0 \) such that the solution
\[ P_{\mu}(\alpha B) = 0. \]

This is the eigenvalue relation for \( \mu \) in terms of \( B \).

Now notice that \( 0 < B < \pi \).

Suppose that \( 0 < \alpha < B \) then \( 0 < B < x < 1 \). The problem (1) is that:

\[
\begin{align*}
\frac{d}{dx} \left[ (1-x^2) \frac{dp}{dx} \right] + \mu (\alpha + 1) p &= 0 & (0 < B < x < 1) \\
p &\text{is finite at } x = 1 \\
p(\alpha B) &= 0.
\end{align*}
\]

This is an eigenvalue problem for \( \mu = \mu(B) \).

Remarks:

1. Notice that \( x = 1 \) is a singular point. One solution is singular near \( x = 1 \) the other is finite as \( x \to 1 \).

2. Consider \( (1-x)(1+x) p'' - 2x p' + \mu (\alpha + 1) p = 0 \).

Let \( \delta : x - 1 \) with \( \delta \) small.

\[-2\delta p'' - 2 p' + \mu (\alpha + 1) p \approx 0 \]

\[\delta^2 p'' + \delta p' - \frac{\mu (\alpha + 1)}{2} \delta p = 0\]

Let \( \delta = \delta_0 \rightarrow q_0 (q_0 - 1) + q_0 = 0 \rightarrow q_0 = 0 \) repeated root.

Thus one solution is regular at the origin, the other diverges logarithmically.

3. If \( \mu = 0 \) is a non-negative integer, then \( P_n(x) \) is a polynomial.

\[ P_0(x) = 1 \quad P_1(x) = \frac{1}{2} (3x^2 - 1) \]
\[ P_2(x) = x \]

This corresponds to where we replace \( P(\alpha B) = 0 \) by \( P \) finite at \( x = 1 \).
WE WRITE \[ (1 - x^2) y' + \mu (\mu + 1) y = 0 \quad (\alpha \leq x < 1). \]

\[ y(1) \text{ finite} \quad y(0) = 0 \quad \text{with} \quad \alpha = (\alpha \leq 1) \]

WE LET \( t = x - 1 \) SO THAT

\[ \Gamma (t^2 + 2t) y' + \mu (\mu + 1) y = 0 \]

WE EXPAND \( y = \sum_{k=0}^{\infty} a_k t^k \)

WE SUBSTITUTE TO FIND THE RECURRENCE RELATION

\[ a_k = \frac{(\mu - (k^2)) (\mu + k)}{2 \ k^2} \ a_{k-1}, \quad k \geq 1 \]

This yields with \( a_0 = 1 \) that

\[ a_k = \frac{(\mu + k) (\mu + k - 1) \ldots (\mu + 2) (\mu + 1) (\mu - 1) \ldots (\mu - (k-1))}{2^k (k^2)^2} \]

In this way we define

\[ P_\mu (x) = \sum_{k=0}^{\infty} \frac{(\mu + k) (\mu + k - 1) \ldots (\mu + 2) (\mu + 1) (\mu - 1) \ldots (\mu - (k-1))}{(2^k) (k^2)} (x-1)^k \]

Which is the Legendre function of the first kind of order \( \mu \).

NOTICE \( a_1 = \frac{\mu (\mu + 1)}{2 (1)^2} \)

\( a_2 = \frac{\mu (\mu + 1) (\mu - 1) (\mu + 2)}{(2 (1)^2) (2 (2)^2)} \)

This series converge for \( |x - 1| < 1 \), i.e. ON THE RANGE \( (\alpha \leq x < 1) \).

Thus, we define \( P_\mu (x) = \sum_{k=0}^{\infty} a_k (\mu) (x-1)^k \) for some \( a_k = a_k (\mu) \)

THE SOLUTION WITH \( y(1) = 1 \) IS \( y = P_\mu (x) \) WITH \( a_0 = 1. \)
Now let's suppose for example that \( \cos B = \frac{1}{2} \), so that \( B = \frac{\pi}{3} \).

Then we use recurrence relation or MATLAB's built-in function to get a plot of \( P_\mu \left( \frac{1}{2} \right) \) versus \( \mu \).

Let \( \cos B = \frac{1}{2} \) so \( B = \frac{\pi}{3} \). We plot \( P_\mu \left( \frac{1}{2} \right) \) versus \( \mu \) and look for the first zero.

\[ P_\mu \left( \frac{1}{2} \right) \]

\[ \mu_1 \approx 1.777, \quad \mu_2 \approx 4.762 \]

Now from tables of special functions, for some \( A > 0 \),

\[ P_\mu \left( \cos \phi \right) \sim A J_0 \left( (2\mu + 1) \sin \phi / 2 \right) \] as \( \mu \to \infty \).

Now if we have \( \mu \to \infty \) then \( (2\mu + 1) \sin B / 2 \sim z_0 \) with \( J_0(z_0) = 0 \) and \( z_0 \) smallest root of \( J_0 \). Thus for \( \mu \to \infty \), which corresponds to \( B \to 0 \), we have using \( \sin B \sim B \) that \( (2\mu + 1) B / 2 \sim z_0 \) or \( \mu \sim z_0 / B - 1/2 \). We know \( z_0 \approx 2.4048 \).
The second relation from Abramowitz-Stegun, p. 335, formula (8.6.20) is that
\[ \rho_{\mu}(c_0q) \sim \rho_0(c_0q) + 2 \log \left( \cos \left( \frac{q}{2} \right) \right) \frac{1}{\mu} \text{ as } \mu \to 0. \] (See remark at end of note.)

Now since \( \rho_0(c_0q) = 1 \) we let \( B \to \bar{\nu} \) to obtain
\[ \cos \left( \frac{q}{2} \right) \sim \cos \left( \frac{\bar{\nu}}{2} \right) - \frac{1}{2} \sin \left( \frac{\bar{\nu}}{2} \right) (B - \bar{\nu}) + \cdots \Rightarrow B \to \bar{\nu}. \]

Thus for \( B \to \bar{\nu} \), \( \rho_{\mu}(c_0q) \sim 1 + 2 \mu \log \left( \frac{B - \bar{\nu}}{2} \right) \) as \( \mu \to 0 \).

Setting \( \rho_{\mu}(c_0q) = 0 \) yields
\[ \mu \sim \frac{1}{2 \log \left( \frac{B - \bar{\nu}}{2} \right)} \]
for \( B \to \bar{\nu} \).

Using \( B = 170^\circ \) we calculate \( \mu \approx 0.2 \).

This gives the following curve after computing \( \rho_{\mu}(c_0q) = 0 \) by Maple.

**Remark.** A re-entrant corner is one for which \( B > \pi/2 \). A lightning rod would have \( B \to \bar{\nu} \). We have \( V \sim A \mu^{\nu} \rho_{\mu}(c_0q) \) so \( V_\mu \sim A \mu^{\nu - 1} \rho_{\mu}(c_0q) \).

As \( B \to \bar{\nu} \) then \( \mu \to 0 \). For \( B > \pi/2 \) we have \( \mu < 1 \) so \( V_\mu \to +\infty \) as \( \mu \to 0 \).
Finally, we calculate electric fields

\[ E_r = -\frac{1}{r}\frac{\partial V}{\partial r}, \quad E_q = -\frac{1}{r}\frac{\partial V}{\partial q}, \]

\[ V = A \Gamma^\mu \mu_{\mu} (\cos q), \]

\[ E_r = -\mu A \Gamma^{-1}_\mu \mu_{\mu} (\cos q), \]

\[ E_q = A \Gamma^{-1}_\mu \sin q \mu_{\mu} (\cos q). \]

\( r \) and \( E_q \) behave like \( r^{-1} \) near the tip.

Hence, if \( 0 < \mu < 1 \rightarrow B > \frac{\pi}{2} \) we have very strong fields near the corner.

**Remark (x)**

\[ Q'' + (\cot q) Q' + \mu (\mu + 1) Q = 0 \]

let \( x = \cos q \) and \( \ddot{Q} = \frac{dQ}{dx} \).

Then

\[ \frac{dQ}{dq} = \ddot{Q} \sin q \]

\[ \frac{d^3 Q}{dq^3} = \dddot{Q} \sin^3 q - \cos q \ddot{Q} \]

so

\[ \dddot{Q} \sin^3 q - \cos q \ddot{Q} + (\cot q) \ddot{Q} + \mu (\mu + 1) Q = 0 \]

yield,

\[ \dddot{Q} \sin^3 q - 2 \cos q \ddot{Q} + \mu (\mu + 1) Q = 0 \]

so

\[ \frac{(1-x^3)\ddot{Q}}{dx} + \mu (\mu + 1) Q = 0. \]
Remark: The Legendre function \( y_i^2(\mu, x) \) is the solution to

\[
(1-x^2) y_i' - \mu (\mu + 1) y_i = 0, \quad x < 1
\]

that satisfies \( y_i(1) = 1 \) (normalization condition).

For \( \mu \to 0 \) we used \( P_{\mu}(\cos \varphi) \sim 1 + 2 \mu \log \left( \cos \left( \frac{\varphi}{2} \right) \right) \) as \( \mu \to 0 \)

when \( 0 < \varphi < \pi \). This formula from Abramowitz and Stegun can be derived directly.

We expand \( y_i = 1 + \mu y_i + \ldots \) for \( \mu \ll 1 \).

Upon substituting and keeping \( O(\mu) \) terms,

\[
(1-x^2) y_i' = -1
\]

\[ y_i(1) = 0. \]

We integrate \( (1-x^2) y_i' = -x + C \)

to eliminate the singularity at \( x=1 \) set \( C = 1 \)

\[ y_i' = \frac{x-1}{x^2-1} \frac{1}{x+1} \]

so \[ y_i = \log |x+1| + B. \]

Put \( y_i(1) = 0 \) \( \to B = -\log 2 \). Thus for \( \mu \ll 1 \),

\[ y \sim 1 + \mu \log \left( \frac{x+1}{2} \right) + O(\mu^2) \text{ on } -1 < x < 1 \]

Now this gives \( P_{\mu}(x) \sim 1 + \mu \log \left( \frac{x+1}{2} \right) \) as \( \mu \to 0 \).

But \( x = \cos \varphi = \cos^2(\varphi/2) - \sin^2 \varphi/2 = 2 \cos^2(\varphi/2) - 1 \). So \( \frac{x+1}{2} = \cos^2(\varphi/2) \)

Upon substituting we get

\[ P_{\mu}(\cos \varphi) \sim 1 + 2 \mu \log \left( \cos \left( \frac{\varphi}{2} \right) \right) \text{ as } \mu \to 0. \]
REMARK THE LEGENDRE FUNCTION CAN ALSO BE EXPRESSED AS AN INTEGRAL.

IN FACT, THE MEHLER-DIRICHLET FORMULA GIVES THE INTEGRAL REPRESENTATION

\[ P_{\mu}(\cos \varphi) = \frac{1}{\pi} \int_{0}^{\pi} \frac{\cos \left( (\mu + 1/2) \varphi \right)}{\sqrt{\cos \theta - \cos \varphi}} \, d\theta \]

BY USING A STATIONARY PHASE APPROXIMATION WE CAN CALCULATE \( P_{\mu}(\cos \varphi \mid \ref{eqn:stationary_phase}) \)

AS \( \mu \to \infty \), THIS INTEGRAL RELATION CAN BE USED FOR COMPUTATIONS.

THE FULL NUMERICAL (HEAVY SOLID CURVE), THE SMALL \( \mu \) APPROXIMATION \( \mu \approx -\frac{1}{2} \log \left( \frac{a - b}{2} \right) \) AND THE LARGE \( \mu \) APPROXIMATION \( \mu \approx \frac{\pi}{B} - \frac{1}{2} \) WITH \( z_0 = 2.4048 \) (DOTTED CURVE) ARE SHOWN BELOW.

\[ \begin{align*}
\mu_{\text{min}} & \approx \frac{1}{2} \left( 1 + \sqrt{1 - 4\pi^2 \frac{a - b}{2}} \right) \\
\beta(\text{degrees}) & \begin{cases} 30 & 45 & 60 & 75 & 90 & 105 & 120 & 135 & 150 & 165 & 180 \\
0.0 & 0.5 & 1.0 & 1.5 & 2.0 & 2.5 & 3.0 
\end{cases}
\end{align*} \]