PROBLEM 1  CONSIDER THE DELAY EQUATION

\[ x'(t) = -ax(t - \pi/2) - x^3(t), \quad a > 0. \]

(i) PROVE THAT \( x = 0 \) IS STABLE WHEN \( 0 < q < 1 \) AND THAT THE
LINEARIZED PROBLEM AROUND \( x = 0 \) HAS A PURE IMAGINARY EIGENVALUE
WHEN \( d = 1 \).

(ii) LET \( d = 1 + \epsilon \) AND \( x = \sqrt{\epsilon} y \) TO OBTAIN THAT \( y(t) \) SATISFIES

\[ y'(t) = -y(t - \pi/2) - \epsilon \left[ y(t - \pi/2) + y^3(t) \right]. \]

NOW USE THE POINCARE-LINDSTEDT METHOD WITH \( \tau = \omega t \)
WHERE \( \omega = 1 + \epsilon \omega_1 + \cdots \) AND LET \( y(t) = y(\tau/\omega) \), TO SHOW THAT

(iii) HAS A PERIODIC SOLUTION WHEN \( \epsilon \ll 1 \) WITH

\[ y \sim \frac{6\pi^2}{3} \cos \left( \omega t \right) + O(\epsilon) \quad \text{with} \quad \omega = 1 + \epsilon + O(\epsilon^2). \]

(iv) NEXT, USE A MULTIPLE-SCALE APPROACH TO SOLVE (ii) BY SEEKING
A SOLUTION IN THE FORM

\[ y(t) = y(t, \tau) = y_0(t, \tau) + \epsilon y_1(t, \tau) + \cdots \quad \text{with} \quad \tau = \epsilon t. \]

SHOW THAT \( y_0 = A(\tau) \sin t + B(\tau) \cos t \) WHERE \( A(\tau) \) AND \( B(\tau) \) SATISFY

\[ A' - \frac{\pi}{2} B' = -B - 3A^3 - 3AB^2, \quad B' + \frac{\pi}{2} A' = A - 3A^2B - 3B^3. \]

WRITE \( A = R \cos \phi \) AND \( B = R \sin \phi \) TO OBTAIN THAT

\[ R' = \frac{(2\pi - 3R^2)R}{\pi^2 + 4}, \quad \phi' = \frac{1}{2} \left( \frac{3R^2\pi + 8}{\pi^2 + 4} \right). \]

DOES THE EQUILIBRIA AGREE WITH THE RESULT IN (ii)?

(iv) SOLVE (ii) NUMERICALLY USING THE DDE PROGRAM IN MATLAB
FOR \( \epsilon = 0.1 \) AND INITIAL DATA \( y(t) = \sin t \) FOR \(-\pi/2 < t < 0\).
ON THE SAME GRAPH PLOT THE AMPLITUDE \( R(\tau) \) WITH \( \tau = \epsilon t \)
FROM THE MULTIPLE SCALE APPROACH IN (iii).
PROBLEM 2 A MODEL FOR THE STABILIZATION OF AN INVERTED PENDULUM WITH DELAY IS

\[ \ddot{\phi}(t) + \Pi \dot{\phi}(t) - g \sin(\phi(t)) + R_0 \phi(t-\tau) = 0 \]

WHERE \( R_0 \phi(t-\tau) \) IS THE "CONTROL" ACTING AS A DELAYED TORQUE.

HERE \( R_0 > 0, \Pi > 0, g > 0 \) AND \( \tau > 0 \). WE WILL CONSIDER \( \Pi \) BIFURCATION PARAMETER, \( R_0 \) AND \( \tau \).

(i) Determine the equilibrium solutions and how the steady state bifurcation diagram changes as \( R_0 \) passes through the value \( g \). What is the stability property of these solutions with no delay (\( \tau = 0 \))? (ii) Prove that the steady-state solution \( \phi_e = 0 \) is unstable for any \( \tau > 0 \) when \( 0 < R_0 < g \).

(iii) Suppose that \( R_0 > g \). Show that the steady-state solution \( \phi_e = 0 \) undergoes a Hopf bifurcation at some critical value \( \tau_H = \tau_H(R_0) \) (Hint: the curve will be obtained implicitly). Show that \( \tau_H \to \pi/2 \) as \( R_0/g \to 1^+ \).

(iv) For \( g = 1 \) and \( \Pi = 2 \) find and numerically plot the region in the \( R_0 \) vs \( \tau \) parameter plane where the inverted state of the pendulum (\( \phi_e = 0 \)) is stable.
PROBLEM 3

Consider the thermostat model for \( U(x, t) \) given by

\[
U_t = k U_{xx} \quad \text{on} \quad 0 < x < L, \quad t > 0
\]

with \( U_x(0, t) = \gamma U(L, t) \) and \( U(L, t) = 0 \)

with initial data \( U(x, 0) \) prescribed \( x = 0 \) \( x = L \)

Here \( k > 0 \), \( \gamma > 0 \) and \( L > 0 \) constants.

(i) Non-dimensionalize this problem to

\[
U_t = U_{xx}, \quad 0 < x < 1, \quad t > 0
\]

\[
U_x(0, t) = b U(1, t), \quad U_x(1, t) = 0
\]

with initial data.

Here \( b > 0 \) is to be found.

(ii) By separating variable in \((+)\) find solution in the form

\[ U(x, t) = e^{-\lambda t} \Phi(x) \]

Derive the eigenvalue problem for \( \lambda \) and \( \Phi(x) \).

(it is not of Sturm-Liouville type).

(iii) Numerically compute the minimum value \( b_r \) of \( b \)

for which a pair of eigenvalues cease to be real-valued.

(iv) Numerically find the minimum value \( b_H \) of \( b \) where a pair of eigenvalues first enters the unstable half-plane \( \text{Re} \lambda < 0 \). Give a rough sketch of what happens to the eigenvalues as \( b \) is varied on \( 0 < b < b_H \).

(remark at \( b = b_H \) an oscillation sets in for the solution \( U(x, t) \).)