Problem 1: Consider the hyperbolic-parabolic system

\[ \frac{\partial P}{\partial t} = M \frac{\partial P}{\partial x} + \frac{1}{\epsilon} F(P) \text{ on } 0 < x < 1 \text{ for } 0 < \epsilon \ll 1, \quad t > 0, \]

where

\[ P = \begin{pmatrix} P^R \\ P^L \\ P^U \end{pmatrix}, \quad M = \begin{pmatrix} -\frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial^3}{\partial x^3} \end{pmatrix} \]

and

\[ F(P) = \begin{pmatrix} \nu_a \Phi(x) g(P^U) - P^R \\ \nu_a (1 - \Phi(x)) g(P^U) - P^L \\ -\nu_a g(P^U) + P^R + P^L \end{pmatrix} \]

where \( g(P^U) = \frac{g_0 P^U}{\nu_a P^U} \) (with \( \nu > 0 \), \( g_0 > 0 \), \( 0 < \Phi(x) < 1 \), and \( \nu_a > 0 \) is constant.

We suppose that the conservation condition

\[ (P^R - P^L - D \frac{\partial P^U}{\partial x}) \bigg|_{x:0,1} = 0 \quad \text{holds.} \]

(i) Show that a quasi-steady-state solution for \( \epsilon \ll 1 \) is given by

\[ P^0(x) = \begin{pmatrix} \Phi(x) \nu_a g(\alpha) \\ (1-\Phi(x)) \nu_a g(\alpha) \end{pmatrix} \text{ for some } \alpha = \alpha(x,t). \]

(ii) By calculating the eigenvalues \( \lambda \) of the Jacobian of \( F(P) \), show that \( \lambda = 0, \lambda = -1 \) and \( \lambda = -1 - \nu_a g(\alpha) \). Conclude that \( P^0(x) \) is a stable slow manifold whenever \( g(P^U) \) is monotone increasing.

(iii) From a perturbation analysis show that \( \alpha(x,t) \) satisfies

\[ \frac{d}{dt} \left( \nu_a g(\alpha) + \alpha \right) = \frac{d}{dx} \left( D \frac{d\alpha}{dx} - (2\Phi(x) - 1) \nu_a g(\alpha) \right) \quad \text{on } 0 \leq x \leq 1, \quad t > 0 \]

with boundary conditions \( \left( D \frac{d\alpha}{dx} - (2\Phi(x) - 1) \nu_a g(\alpha) \right) = 0 \) at \( x=0,1 \).
PROBLEM 2 (ENZYME KINETICS) (Michaelis-Menten Reaction)

A substrate $S$ is converted into a product $P$ by mean of an enzyme $E$

According to

\[ S + E \overset{k_1}{\underset{k_{-1}}{\rightleftharpoons}} SE \overset{k_2}{\rightarrow} E + P \]

Let $c_1, c_2, c_3, c_4$ be concentration of $S, E, SE,$ and $P$.

(i) Use law of mass action to show that

\[ \dot{c}_1 = -k_1 c_1 c_3 + k_{-1} c_2 \]
\[ \dot{c}_2 = -k_3 c_1 c_2 + (k_{-1} + k_2) c_3 \]
\[ \dot{c}_3 = k_1 c_1 c_2 - (k_{-1} + k_2) c_3 \]
\[ \dot{c}_4 = k_2 c_3 \]

(ii) Assume $c_1(0) = a_1, c_2(0) = a_2, c_3(0) = c_4(0) = 0$.

Derive that

\[ \dot{c}_1 = -k_1 c_1 c_2 + (k_1 c_1 + k_{-1}) c_3 \]
\[ \dot{c}_3 = k_1 c_1 c_2 - (k_1 c_1 + k_{-1} + k_2) c_3 \]
\[ \dot{c}_1(0) = a_1 \]
\[ \dot{c}_3(0) = 0 \]

(iii) Let $x = c_3/a_2, y = c_3/a_1$, and introduce a non-dimensional time $t = wT$, for some $w$ to get

\[ \dot{y} = -y + \left(\frac{y + \frac{k_{-1}}{k_1 a_1}}{\frac{k_{-1}}{k_1 a_1}}\right) x, \quad y(0) = 1 \]
\[ \frac{a_2}{a_1} \begin{cases} \dot{x} = y - \left(\frac{y + \frac{k_{-1} + k_2}{k_1 a_1}}{\frac{k_{-1} + k_2}{k_1 a_1}}\right) x, \quad x(0) = 0 \end{cases} \]
(iii) Assume that $\varepsilon = q_2/q_1 \ll 1$. Show that (x) has a stable slow manifold and that on this manifold

\[
\dot{y} = -\frac{A}{y+\mu} y, \quad y(0) = 1
\]

for $A = \frac{K_2}{K_1 q_1}, \quad \mu = \frac{K_1 + K_2}{K_1 q_1}$.

How does $y \to 0$ as $t \to \infty$?

Is the initial point $x(0) = 0, y(0) = 1$ on the stable manifold at $t = 0$?

(iv) Plot the numerical solution to (x) for $K_1 = K_2 = K_0 = 1$ with $q_2 = 0.5$, compare the numerical with the solution to (4).
Problem 3 Consider the four-state Markov chain

\[ \begin{array}{c}
P_2 \xrightarrow{\epsilon} P_1 \xrightarrow{\epsilon} P_1 \xrightarrow{1} P_2 \\
\downarrow \quad \downarrow \quad \downarrow
\end{array} \]

(i) Let \( P = \begin{pmatrix} p_2 \\ p_1 \\ p_1 \\ p_2 \end{pmatrix} \). Argue that the DDE system is

\[ P' = \lambda_\epsilon P \quad \lambda_\epsilon = \begin{pmatrix} -\epsilon & 1 & 0 & 0 \\
\epsilon & -1-\epsilon & \epsilon & 0 \\
0 & \epsilon & -1-\epsilon & \epsilon \\
0 & 0 & 0 & 1-\epsilon \end{pmatrix} \]

(Assume \( \epsilon \ll 1 \))

(ii) Writing \( \lambda_\epsilon = \lambda_0 + \epsilon \lambda_1 \), show that \( \lambda_0 \) has eigenvalues \( \lambda = 0 \) (multiplicity 2) and \( \lambda = -1 \) (multiplicity 2).

(iii) Show that the slow vectors are \( \phi_1 = (0, 0, 0, 1)^T \), \( \phi_2 = (1, 0, 0, 0)^T \), and that the null vectors in the adjoint of \( \lambda_0 \) are \( \psi_1 = (0, 0, 1, 1)^T \), \( \psi_2 = (1, 1, 0, 0)^T \).

(iv) What is the \( O(\epsilon) \) approximation to the solution on the slow manifold?

(v) By going to one higher order show that the slow manifold is characterized by

\[ (xy) \quad S_2 \approx \epsilon^2 (S_+ - S_-) \]

(vi) By finding the eigenvalues of \( \lambda_\epsilon \) explicitly for \( \epsilon \to 0 \), argue that \((xy)\) is correct.
PROBLEM 1

(i) We set $f = 0$ to get $f_i = 0, \ j = 1, \ldots, 3$, which yields

$$\begin{align*}
    \lambda a \phi(x) g(p^L) - p^R = 0, \\
    \lambda a (1 - \phi(x)) g(p^U) - p^L = 0, \\
    -\lambda a g(p^U) + p^R + p^L = 0.
\end{align*}$$

Since $f_1 + f_2 + f_3 = 0$, there is a free variable which we choose as

$$p^U = q(x, t).$$

Then solving for $p^L$ and $p^R$ we obtain the manifold $\mathcal{P}^0(x)$ given by

$$\begin{align*}
    \mathcal{P}^0(x) = \begin{pmatrix}
        \lambda a \phi(x) g(x) \\
        (1 - \phi(x)) \lambda a g(x) \\
        q
    \end{pmatrix}.
\end{align*}$$

(ii) Now we check that we can use this as a slow manifold.

In

$$p_t = Mp + \frac{1}{\epsilon} f(p),$$

we let $\mu = t/\epsilon$ so that $\mu_t = \mu_\epsilon + 1/\epsilon$ and to leading order if $\mu_\epsilon(x, \epsilon) = \mu(x, \epsilon \tau)$ then $\mu_\epsilon = f(\mu)$. We observe that

$f(\mathcal{P}^0(x)) = 0$ so that $\mu = \mathcal{P}^0(x)$ is an equilibrium solution for the dynamics. Now linearizing around $\mathcal{P}^0(x)$ we write

$$\mu = \mathcal{P}^0(x) + \tilde{\mu}$$

with $\|\tilde{\mu}\| < 1$ and obtain

$$\tilde{\mu}_t = J \tilde{\mu}$$

where $J$ is the Jacobian of $f$ evaluated at $\mathcal{P}^0(x)$. We say that $\mathcal{P}^0(x)$ is a slow manifold if the eigenvalues $\lambda$ of $J$ satisfy $\Re \lambda < 0$.

Now we calculate the matrix $J$. We get that
\[
J = \begin{pmatrix}
-1 & 0 & \nu_0 P(x) g'(x) \\
0 & -1 & \nu_0 (1 - P(x)) g'(x) \\
1 & 1 & -\nu_0 g'(x)
\end{pmatrix}
\]

Now
\[
\det(J - \lambda I) = \det
\begin{pmatrix}
-1 - \lambda & 0 & \nu_0 P(x) g'(x) \\
0 & -1 - \lambda & \nu_0 (1 - P(x)) g'(x) \\
1 & 1 & -\nu_0 g'(x) - \lambda
\end{pmatrix}
\]

We calculate this as
\[
\det(J - \lambda I) = 0 \Rightarrow
- (1 + \lambda) \left[ (1 + \lambda) [ \lambda + \nu_0 g'(x)] - \nu_0 (1 - P(x)) g'(x) \right] + (1 + \lambda) \nu_0 P(x) g'(x) = 0
\]

so
\[
- (1 + \lambda) \left[ (1 + \lambda) [ \lambda + \nu_0 g'(x)] - \nu_0 (1 - P(x)) g'(x) - \nu_0 P(x) g'(x) \right] = 0
\]

\[
- (1 + \lambda) \left[ \lambda + (1 + \lambda) \nu_0 g'(x) - \nu_0 g'(x) \right] = 0
\]

\[
- (1 + \lambda) \left[ \lambda + \nu_0 g'(x) + 1 \right] = 0
\]

Thus yield that
\[
\lambda = 0, \quad \tilde{\lambda} = -1, \quad \hat{\lambda} = -1 - \nu_0 g'(x).
\]

Now \(g(x) = \frac{g_m x}{1 + \alpha}\) is monotone increasing so that
\(g'(x) > 0\). Thus \(\Re \lambda \leq 0\) for all eigenvalues of the Jacobian, and \(P^0(x)\) is a slow manifold.

Starting from some initial condition \(\hat{z}^0(x)\), the ODE \(\frac{\partial}{\partial \tau} \hat{z} = \tilde{F}(\hat{z})\) will approach \(P^0(x_0)\) for some \(x_0\).

As \(\tau \to \infty\).
Now we return to
\[ p_t = M p + \frac{1}{\varepsilon} f(p) \]
We expand \( p = p^0(x) + \varepsilon p^1 + \ldots \) and derive that
\[ d_t p^0 = M p^0 + J p^1 \] so that
\[ J p^1 = d_t p^0 - M p^0 \tag{4} \]
Since \( J \) has a zero eigenvalue so must \( J^T \). We have
\[ \dim \ker J = 1 \quad \implies \quad \dim \ker (J^T) = 1 \quad \text{and} \]
\[ J^T = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ \eta_q \overline{\tau}(x) & \eta_q (1 - \overline{\tau}(x)) g'(x) & -\eta_q g'(x) \end{pmatrix} \]
We observe that \( \psi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) satisfies \( J^T \psi = 0 \).

Thus from (4) the solvability condition gives
\[ \psi^T d_t p^0 = \psi^T M p^0 \], which will yield a scalar PDE for \( p(x,t) \). Using \( p^0(x) \) from Part (i), we get with
\[ M = \begin{pmatrix} -\frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{d^2}{\partial x^2} \end{pmatrix} \]
\[ p(x) \eta_q g'(x) d_t + (1 - \rho(x)) \eta_q g'(x) \alpha_t + \delta_t = \psi^T d_t p^0 \]
which yields
\[ \psi^T d_t p^0 = \begin{pmatrix} \alpha_t + \eta_q g'(x) \end{pmatrix} \]

Then \( \psi^T M \rho^o = (1 \quad 1) \begin{pmatrix} -\partial_x (\mu_0 \rho g(x)) + \partial_x (1 - \rho) \mu_0 g(x) + D \partial_x^2 \alpha \end{pmatrix} \)
\[ = D \partial_x^2 \alpha + \partial_x \left[(1 - 2 \rho) \mu_0 g(x)\right] \]

As such, we have from setting \( \psi^T \partial_t \rho^o = \psi^T M \rho^o \) that
\[ \left( \alpha + \mu_0 g(x) \right)_t = \partial_x \left[ D \frac{\partial \rho}{\partial x} + (1 - 2 \rho(x)) \mu_0 g(x) \right] \]

which is a nonlinear convection-diffusion equation for \( \alpha(x,t) \).

The conservation of mass condition
\[ \left. \left( \rho^u - \rho^l \cdot \frac{\partial \rho^u}{\partial x} \right) \right|_{x=0,1} = 0 \] condition

in terms of \( \alpha(x,t) \) yields
\[ D \frac{\partial \alpha}{\partial x} - (2 \rho(x) - 1) \mu_0 g(x) = 0. \]
PROBLEM 2 (ENZYME KINETICS)

ENZYME CATALYZED REACTIONS. SUBSTRATE CONVERTED INTO A PRODUCT P BY MEAN OF AN ENZYME E.

\[ S + E \xrightarrow{K_i} SE \xrightarrow{K_2} E + P \]

THE ENZYME IS RECOVERED AFTER CONVERSION TO PRODUCT P.

LAW OF MAJ ACTION: RATE OF CONCENTRATION CHANGE IS PROPORTIONAL TO CONCENTRATION OF SPECIE IN THE REACTION.

i) LET \( c_1, c_2, c_3, c_4 \) BE CONCENTRATION OF S, E, SE AND P. THEN,

\[
\begin{align*}
    c_1' &= -k_1 c_1 c_2 + k_{-1} c_3, & c_1(0) &= q_1, \\
    c_2' &= -k_1 c_1 c_2 + (k_{-1} + k_2) c_3, & c_2(0) &= q_2 \\
    c_3' &= k_1 c_1 c_2 - (k_{-1} + k_2) c_3, & c_3(0) &= 0 \\
    c_4' &= +k_2 c_3, & c_4(0) &= 0
\end{align*}
\]

ii) NOTICE \( \frac{d}{dt} (c_2 + c_3) = 0 \) \( \frac{d}{dt} (c_1 + c_3 + c_4) = 0 \)

THUS WE HAVE TWO CONSTANT OF MOTION

\[ a_2 = c_2 + c_3, \quad c_1 = c_1 + c_3 + c_4. \]

WITH \( c_2, c_1 \) INDEPENDENT OF \( t \). SUBSTITUTE \( c_2 = a_2 - c_3 \) INTO \( c_1, c_3 \) EQUATIONS.

THUS

\[
\begin{align*}
    c_1' &= -k_1 c_1 c_2 + (k_{-1} c_1 + k_1) c_3, \\
    c_3' &= k_1 c_1 c_2 - (k_{-1} c_1 + k_2) c_3
\end{align*}
\]

WHICH ARE 2 ODES FOR \( c_1 \) AND \( c_3 \).
If we will assume that the initial concentration $c_3$ of $S$ is high relative to that of the enzyme, hence quickly in time the enzyme should exist only in the complex $E_S$ and $c_3$ should then equilibrate, or reach a steady-state, as little free enzyme is available.

Our limit of interest is $q_1/q_2 \gg 1$.

We let $t = \omega t$, $x = c_3/q_2$, $y = c_1/q_1$.

Then
\[
\frac{1}{\omega} \frac{\dot{q}_1}{q_1} \frac{\dot{y}}{y} = -k_1 \frac{a_1}{q_1} \frac{a_2}{q_2} \frac{a_1}{q_1} y \left( \frac{k_1}{q_1} a_1 y + k_{-1} \right) q_2 x
\]

and
\[
\frac{1}{\omega} \frac{\dot{q}_2}{q_2} \frac{\dot{x}}{x} = \frac{k_1}{q_1} a_1 q_2 y \left( k_1 a_1 y + k_{-1} + k_2 \right) q_2 x
\]

Then we let
\[
\frac{1}{\omega} \frac{\dot{y}}{y} = -y + \left( \frac{k_1}{q_1} a_1 y + k_{-1} \right) x
\]

and
\[
\frac{1}{\omega} \frac{\dot{x}}{x} = y - \left( k_1 a_1 y + k_{-1} + k_2 \right) x
\]

Choose $\omega = \frac{1}{k_1 a_2}$ so that
\[
\begin{align*}
\dot{y} &= -y + \left( y + \frac{k_{-1}}{k_1 a_1} \right) x, \quad y(0) = 1 \\
\frac{a_2}{a_1} \dot{x} &= y - \left( y + \frac{k_{-1} + k_2}{k_1 a_1} \right) x, \quad x(0) = 0
\end{align*}
\]
\( \gamma = \frac{q_2}{q_1} \)

\[
\dot{y} = -y + \left( y + \frac{K_1}{K_1 + K_2} \right) \frac{y}{y+\mu} \]

\[
\dot{x} = y - \left( y + \frac{K_1 + K_2}{K_1 + q_1} \right) x = f(x, y)
\]

Define \( \Delta = \frac{K_2}{K_1 + q_1} \quad \mu = \frac{K_1 + K_2}{K_1 + q_1} \)

Then

\[
\begin{align*}
\dot{y} &= -y + \left( y + \mu - \Delta \right) x = g(x, y) \\
\dot{x} &= y - \left( y + \mu \right) x = f(x, y)
\end{align*}
\]

The slow dynamics is \( f = 0 \) which gives \( x = \frac{y}{y+\mu} \) so \( x^*(y) \).

Now \( f_x(x^*, y) = -\left( y + \mu \right) < 0 \) so stable slow manifold.

Now, on the slow manifold

\[
\dot{y} = -y + \left( y + \mu - \Delta \right) \frac{y}{y+\mu} = -\frac{\Delta y}{y+\mu}
\]

Thus dynamics on slow manifold is

\[
\begin{align*}
\dot{y} &= -\frac{\Delta y}{y+\mu} \quad y(0) = 1 \\
\end{align*}
\]

So \( y \to 0 \) as \( t \to +\infty \).

As \( y \to 0 \), we have \( \dot{y} = -\frac{\Delta y}{\mu} \) so \( y \sim c e^{-\Delta t/\mu} \) as \( t \to \infty \).
(iv) For $K_1 = K_2 = q_1 = 1$ and $q_2/q_1 = \varepsilon = 0.05$,
we have the full system
\[
\begin{align*}
\dot{y} &= -y + (y + 1)x, \quad y(0) = 1 \\
\varepsilon \dot{x} &= y - (y + 2)x, \quad x(0) = 0
\end{align*}
\]
and the slow manifold approximation
\[
\dot{y} \approx \frac{-y}{y+2}, \quad y(0) = 1
\]
which is separable and yields
\[
\frac{t}{\varepsilon} = 1 - \frac{y - 2 \log y}{y+2}
\]
For $\varepsilon = 0.05$ our comparison of the dynamics of the full system and the slow manifold show that the slow manifold approximation is very accurate.
PROBLEM 3

(i) IT IS CLEAR FROM THE DIAGRAM THAT $P' = \lambda \cdot P$

WHERE

$\lambda = \begin{pmatrix}
-3 & 1 & 0 & 0 \\
3 & -1 & -3 & 0 \\
0 & 3 & -1 & -3 \\
0 & 0 & 1 & -3
\end{pmatrix}$

(ii) WE DECOMPOSE

$\lambda = \lambda_0 + \epsilon \lambda_1$ WHERE

$\lambda_0 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$

$\lambda_1 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}$

NOW

$\det(\lambda_0 - \lambda_0 I) = \det \left( \begin{pmatrix}
-\lambda & 1 & 0 & 0 \\
0 & -1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & -1
\end{pmatrix} \right) = -\lambda \det \left( \begin{pmatrix}
0 & -1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix} \right) - 1 \det \left( \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix} \right)$

THUS YIELDS THAT

$\det(\lambda_0 - \lambda_0 I) = -\lambda [-1(\lambda + 1)]^2 - 1 (0) = \lambda^2(\lambda + 1)^2$

THUS $\lambda = 0$ (REPEATED) AND $\lambda = -1$ (REPEATED).

(iii) NOW FOR $\lambda = 0$ WE WANT TO FIND KERNEL $\lambda_0$.

THIS IS EASY BY INSPECTION $\lambda_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$, $\lambda_0 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$.

THUS

KERNEL $\lambda_0 = \text{span} \{ \phi_1, \phi_2 \}$

$\phi_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\phi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

NOW THE ADJOINT PROBLEM IS WHAT INNER PRODUCT $\langle \psi, \nu \rangle$:

$\lambda_0^\ast = \lambda_0^T = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$

BY INSPECTION

$\lambda_0^\ast \psi_1 = 0$, $\lambda_0^\ast \psi_2 = 0$ with

$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\psi_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

THUS

KERNEL $\lambda_0^\ast = \text{span} \{ \psi_1, \psi_2 \}$. 


Now we expand for long time. Let $\tau = \epsilon t$ so $t = O(1/\epsilon)$ implies $\tau = O(1)$.

We have $\epsilon \dot{P} = \left[ A_0 + \epsilon A_1 \right] P$.

We write $P = [ S_1(\tau) \phi_1 + S_2(\tau) \phi_2 ] + \epsilon P_1 + \epsilon^2 P_2 + \cdots$

where $P_0 = S_1(\tau) \phi_1 + S_2(\tau) \phi_2$.

We have $\epsilon \dot{P}_0 + \epsilon^2 \dot{P}_1 = A_0 P_0 + \epsilon \dot{A}_0 P_0 + \epsilon^2 \dot{A}_0 P_1 + \epsilon \dot{A}_1 P_0 + \epsilon^2 \dot{A}_1 P_1$

Thus gives equations of power of $\epsilon$:

$O(1)$ \quad $A_0 P_0 = 0 \rightarrow \dot{P}_0 = S_1 \phi_1 + S_2 \phi_2$

$O(\epsilon)$ \quad $\dot{A}_0 P_1 = \dot{P}_1 + \dot{A}_1 P_0$

$O(\epsilon^2)$ \quad $\dot{A}_0 P_2 = \dot{P}_2 + \dot{A}_1 P_1$

Now the solvability condition for $O(\epsilon)$ is

$\psi_j^T A_0 P_1 = (\psi_j^T \psi_i) P_1 = (\psi_j^T \psi_i) P_1 = \psi_j^T \dot{P}_0 - \psi_j^T \dot{A}_j P_0 = 0 \quad j = 1, 2$

since $\dot{A}_j^T \psi_j = 0$ for $j = 1, 2$.

Thus yields with $P_0 = S_1 \phi_1 + S_2 \phi_2$ that

\begin{align*}
(\psi_1^T \phi_1) \dot{S}_1 + (\psi_1^T \phi_2) \dot{S}_2 &= S_1 \psi_1^T \dot{A}_1 \phi_1 + S_2 \psi_1^T \dot{A}_1 \phi_2 \\
(\psi_2^T \phi_1) \dot{S}_1 + (\psi_2^T \phi_2) \dot{S}_2 &= S_1 \psi_2^T \dot{A}_1 \phi_1 + S_2 \psi_2^T \dot{A}_1 \phi_2
\end{align*}

We now calculate the term in the ODE $j$

$\psi_j^T \phi_1 = 0, \quad \psi_j^T \phi_2 = 1, \quad \psi_j^T \phi_1 = 1, \quad \psi_j^T \phi_2 = 0.$
Thus, (x) becomes

\[
\begin{align*}
\dot{S}_2 &= S_1 \, \omega_1^T \, \lambda, \phi_1 + S_2 \, \omega_2^T \, \lambda, \phi_2 \\
\dot{S}_1 &= S, \, \omega_2^T \, \lambda, \phi_1 + S_2 \, \omega_2^T \, \lambda, \phi_2
\end{align*}
\]

Now

\[
\begin{align*}
\lambda, \phi_1 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \omega_2^T \, \lambda, \phi_1 = 0 \\
\lambda, \phi_2 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \omega_2^T \, \lambda, \phi_2 = 0
\end{align*}
\]

Thus, we obtain from \((xy)\) that in fact \(\dot{S}_1 = \dot{S}_2 = 0\)!

This shows that the slow dynamics is stationary on the long time scale \(T = \varepsilon t\). To determine the dynamics we must proceed to a longer time scale \(T = \varepsilon^2 t\), so \(t = O(1/\varepsilon^2) \Rightarrow T = O(1)\).

Notice that the key issue is that \(\lambda_0, \phi_2 = -\lambda_1, \phi_2\) is in fact solvable since \(\omega_2^T \lambda, \phi_j = 0\) for \(j = 1, 2\).

Modified expansion let \(T = \varepsilon^2 t\) and \(* = d/dT\) now.

Then

\[\varepsilon^3 \dot{P} = \lambda_0 \dot{P} + \varepsilon \lambda_1 \dot{P}\]

We expand \(\dot{P} = \dot{P}_0 + \varepsilon \dot{P}_1 + \varepsilon^2 \dot{P}_2 + \ldots\)

To obtain

\[\begin{align*}
(1) \quad \lambda_0 \dot{P}_0 &= 0 \quad \Rightarrow \quad \dot{P}_0 = S_1(T) \phi_1 + S_2(T) \phi_2 \\
(2) \quad \lambda_0 \dot{P}_1 &= -\lambda_1 \dot{P}_0 \\
(3) \quad \lambda_0 \dot{P}_2 &= \dot{P}_0 - \lambda_1 \dot{P}_1
\end{align*}\]
Now the problem for $\tilde{P}_1$ is solvable up to adding kernel $(\Lambda_0)$.

The solvability condition for (3) yields

(4) $\vec{w}_j^T \tilde{P}_0 = \vec{w}_j^T \Lambda \tilde{P}_1$ for $j = 1, 2$.

We notice that if $\tilde{P}_1 = P_{1e} + a_1 \phi_1 + a_2 \phi_2$

Then $\vec{w}_j^T \Lambda \tilde{P}_1 = \vec{w}_j^T \left[ \Lambda, P_{1e} + a_1 \phi_1 + a_2 \phi_2 \right] = \vec{w}_j^T \Lambda \tilde{P}_{1e}$

Since $\vec{w}_j^T \Lambda \phi_i = 0$ for $j, i = 1, 2$.

Thus, in solving for $\tilde{P}_1$ we need only normalize $P_{1e}$ to remove its kernel however is convenient. The normalization will not influence the dynamics.

To solve for $\tilde{P}_1$ we write

$\Lambda_0 \tilde{P}_1 = -S, \Lambda, \phi_1 - S, \Lambda, \phi_2$

We decompose $\tilde{P}_1 = -S, \tilde{V}_1 - S, \tilde{V}_2$ so that

$\Lambda_0 \tilde{V}_1 = \Lambda, \phi_1 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

$\Lambda_0 \tilde{V}_2 = \Lambda, \phi_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

We now calculate $\tilde{V}_1$ and $\tilde{V}_2$ explicitly. Notice that if we multiply by $\Lambda_0$ we have

$\Lambda_0 \tilde{V}_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tilde{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \tilde{V}_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$
\[ A_0 V_2 : \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \rightarrow V_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \]

Thus \[ P_1 = -S_1 V_1 - S_2 V_2 \] with \( V_1 = (0, 0, -1, 0)^T \), \( V_2 = (0, -1, 0, 0)^T \).

Now we substitute into (4). Since the left side of (4) will be the same as in (4) we get

\[ \begin{cases} \dot{S}_2 = -S_1 \psi_2^T \phi_1, V_1 - S_2 \psi_1^T \phi_1, V_2 \\ \dot{S}_1 = -S_1 \psi_2^T \phi_1, V_1 - S_2 \psi_2^T \phi_1, V_1 \end{cases} \]

We calculate \[ A_1 V_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \rightarrow \psi_2^T \phi_1, V_1 = 1 \]

\[ A_1 V_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \rightarrow \psi_2^T \phi_1, V_2 = -1 \]

Thus we obtain from (4) the slow dynamics

\[ \begin{cases} \dot{S}_2 = S_1 - S_2 \\ \dot{S}_1 = -S_1 + S_2 \end{cases} \]

We can write this as \( \dot{S}_1 = S_2 - S_1 \), which is what we are requested to show.

In summary, we have that the slow dynamics is

\[ P \sim \phi_1 \psi_1 \phi_1^T + \phi_2 \psi_2 \phi_2^T \] where \( \dot{s}_1/dt = \varepsilon^2 (S_2 - S_1) \)

\[ \dot{s}_1/dt = \varepsilon^2 (S_1 - S_2) \] and \( \phi_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \phi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). Thus there is a slow interchange between \( P \) and \( P_1 \) for \( t \rightarrow \infty \).
This suggests that there is a slow exchange between states \( P_2 \) and \( P_{-2} \) for \( t = O(1/\varepsilon^2) \).

We calculate
\[
\det \left( S - \sigma I \right) = 0 \quad S = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}
\]

so we have \((1-\sigma)(1-\sigma) - 1 = 0\) to \( \sigma^2 + 2\sigma = 0 \) or \( \sigma = 0, \sigma = -2 \).

Thus inserting the \( \varepsilon^1 \) we conclude that the eigenvalues of the slow dynamics are \( \Lambda = -2\varepsilon^1 \) and \( \Lambda = 0 \).

Thus
\[
S = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2\varepsilon^1 t} \quad \text{with} \quad S = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}
\]

we conclude for any initial condition that \( S \rightarrow c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) as \( t \rightarrow \infty \) for \( t = O(1/\varepsilon^2) \).

We conclude that
\[
P \rightarrow c_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \quad \text{as} \quad t \rightarrow \infty \quad \text{for} \quad t = O(1/\varepsilon^2)
\]

We conclude that there is an "equi-partition" between states \( P_2 \) and \( P_{-2} \).

We now confirm with full numeric. We set \( \varepsilon = 0.05 \) with \( P(0) = (1.75, 0.50, 0.25) \)

for \( P' = \frac{d}{dt} P \).

Notice that indeed \( P_2 \) and \( P_{-2} \) approach a common value on a time scale \( t = O(\varepsilon^2) \approx 400 \).
Now we calculate the eigenvalues of the full system.

We want to show that 3 eigenvalues with $A \approx 2 \varepsilon^1$ and $A \ll O(\varepsilon^1)$.

We now have

$$p(\lambda) = \det (A - \lambda I) = \det \left( \begin{array}{ccc} -\varepsilon - A & 1 & 0 \\ \varepsilon & -(1 + \varepsilon A) & \varepsilon \\ 0 & \varepsilon & -(1 + \varepsilon A) \end{array} \right)$$

We observe trace $(A_\varepsilon) = -2 - 4 \varepsilon$ of eigenvalue.

$$\det (A_\varepsilon) = 0 = \text{product of eigenvalue}.$$ 

Thus, $\lambda = 0 \forall \varepsilon$. Now we calculate

$$p(\lambda) = -(\varepsilon + A) \det \left( \begin{array}{ccc} -(1 + \varepsilon A) & \varepsilon & 0 \\ \varepsilon & -(1 + \varepsilon A) & \varepsilon \\ 0 & 1 & -(\varepsilon + A) \end{array} \right) - \det \left( \begin{array}{ccc} \varepsilon & \varepsilon & 0 \\ 0 & -(1 + \varepsilon A) & \varepsilon \\ 0 & 1 & -(\varepsilon + A) \end{array} \right)$$

$$= - (\varepsilon + A) \left[ (1 + \varepsilon A)(1 + \varepsilon A) - \varepsilon \right] - \varepsilon \left[ -(\varepsilon A) \right]$$

$$= (\varepsilon + A)^2 (1 + \varepsilon A) - \varepsilon (\varepsilon + A)(1 + \varepsilon A) - \varepsilon^2 (\varepsilon + A)^2 - \varepsilon (\varepsilon + A)(1 + \varepsilon A)$$

$$= (\varepsilon + A)^2 (1 + \varepsilon A) - \varepsilon^2 (\varepsilon + A)^2 + \varepsilon^2 - 2 \varepsilon (\varepsilon + A)(1 + \varepsilon A)$$

$$p(\lambda) = (\varepsilon + A)^2 \left[ (1 + \varepsilon A)^2 - \varepsilon^2 \right] + \varepsilon^2 - 2 \varepsilon (\varepsilon + A)(1 + \varepsilon A)$$

$$= (\varepsilon + A)^2 \left[ (1 + \varepsilon A)^2 + 2 \varepsilon (1 + \varepsilon A) \right] + \varepsilon^2 - 2 \varepsilon (\varepsilon + A)(1 + \varepsilon A)$$

$$p(\lambda) = (1 + \varepsilon)^2 (\varepsilon + A)^2 - 2 (1 + \varepsilon + \varepsilon A) \varepsilon (\varepsilon + A) + \varepsilon^2 \left[ 1 - (\varepsilon + A)^2 \right]$$

$$= (1 + \varepsilon + \varepsilon)^2 (\varepsilon + A)^2 - 2 \varepsilon (\varepsilon + A)(1 + \varepsilon A) + \varepsilon^2 \left[ 1 - (\varepsilon + A)^2 \right] \left[ 1 + \varepsilon A \right]$$

$$p(\lambda) = (1 + \varepsilon)^2 \left[ (1 + \varepsilon + \varepsilon)(1 + \varepsilon A)^2 - 2 \varepsilon (\varepsilon + A) + \varepsilon^2 - \varepsilon^2 (\varepsilon + A) \right]$$
Now notice that $\lambda=0$ is a factor of $\varepsilon$ in previous equation (1). Thus we can simplify the expression.

\[ p(\lambda) = (\lambda + 1 + \varepsilon) \left[ (\lambda + \varepsilon)^2 (\lambda + \varepsilon)(1 + \lambda + \varepsilon) - 2 \varepsilon^2 + \varepsilon^2 (\lambda + \varepsilon) \right] \]

\[ = (\lambda + 1 + \varepsilon) \left[ (\lambda + \varepsilon)^2 (\lambda + \varepsilon)(1 + \lambda + \varepsilon) - 2 \varepsilon^2 \right] + \varepsilon^2 \]

\[ = (\lambda + 1 + \varepsilon) \left[ (\lambda + \varepsilon)^2 (1 + 2 \varepsilon + 1) \lambda - \varepsilon \right] + \varepsilon^2 \]

Thus simplified to

\[ p(\lambda) = (\lambda + 1 + \varepsilon) \cdot \lambda \cdot \left[ \lambda^2 + (3 \varepsilon + 1) \lambda + 2 \varepsilon^2 \right] \]

Thus $\lambda_3 = -1 - \varepsilon$, $\lambda_4 = 0$ and $\lambda^2 + (3 \varepsilon + 1) \lambda + 2 \varepsilon^2 = 0$.

$\lambda_1 + \lambda_2 = -3 \varepsilon - 1$.

Notice $\sum \lambda_i = -1 - \varepsilon - 3 \varepsilon - 1 = -2 - 4 \varepsilon = -\text{trace } \lambda$.

Now solving quadratic

\[ \lambda = \frac{-(3 \varepsilon + 1) \pm \sqrt{(3 \varepsilon + 1)^2 - 8 \varepsilon^2}}{2} \]

\[ \lambda_+ = \frac{-(3 \varepsilon + 1) + \sqrt{\varepsilon^2 + 6 \varepsilon + 1}}{2} \]

Notice as $\varepsilon \to 0$, $\lambda_+ \sim \left(1 + 3 \varepsilon\right) - \frac{[1 + 3 \varepsilon]}{2} \sim 1 - 3 \varepsilon$.

Now for $\lambda_+$ we have $\sqrt{1 + h} \sim 1 + h/2 - h^2/8$.

So $\sqrt{1 + (6 \varepsilon + \varepsilon^2)} \sim 1 + \frac{1}{2} (\varepsilon^2 + 6 \varepsilon) - \frac{1}{8} (36 \varepsilon^2) \sim 1 + 3 \varepsilon - 4 \varepsilon^2$.

Thus $\lambda_+ \sim \left(1 + 3 \varepsilon\right) + \frac{[1 + 3 \varepsilon - 4 \varepsilon^2]}{2} \sim -2 \varepsilon^2$. This was what was obtained in our slow-fast analysis.